Worksheet #1 in Geometric Algebra

Clifford represented the three dimensions of physical space with the algebraic elements e_1, e_2 and e_3, where $e_1^2 = e_2^2 = e_3^2 = 1$ and with each element anticommuting, that is $e_j e_k = -e_k e_j$, for $j \neq k$. We also define the trivector $i = e_1e_2e_3$, which allows us to write $e_2e_3 = ie_1$, $e_3e_1 = ie_2$ and $e_1e_2 = ie_3$.

EXERCISES

Vectors in three dimensions

1. Given two vectors $\mathbf{a} = 3e_1 + 4e_3$ and $\mathbf{b} = e_1 + 2e_2 + e_3$, expand their product $\mathbf{ab}$.

(a) From this calculation read of $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$.

(b) What is the value of the cross product (Answer must be a vector, see note below).

(c) Calculate $\mathbf{a}^2$ and hence find the length $|\mathbf{a}|$ of the vector $\mathbf{a}$.

(d) What is the inverse of the vectors $\mathbf{a}$ and $\mathbf{b}$? Check that they are the correct inverses by direct calculation.

(e) Divide the vector $\mathbf{b}$ by the vector $\mathbf{a}$, that is calculate $\mathbf{b}/\mathbf{a}$.(Hint use $\mathbf{a}^{-1}$)

Note: The product of two vectors in three dimensions is

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b}. \tag{1}$$

We can see from Eq. (1), that the square of a vector $\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$, and is a scalar quantity. Hence the Pythagorean length of a vector is simply $|\mathbf{a}| = \sqrt{\mathbf{a}^2}$, and so we can find the inverse vector $\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a}^2}$.

Multivectors

We can combine all algebraics quantities into a multivector to form $M = a + v_1e_1 + v_2e_2 + v_3e_3 + b_1e_2e_3 + b_2e_3e_1 + b_3e_1e_2 + ce_1e_2e_3 = a + \mathbf{v} + i\mathbf{b} + ic$, where $i = e_1e_2e_3$. Calculate the following expressions:

(a) $(2 + e_1 + 3e_2)e_2$

(b) $(1 + e_1)(1 - e_1)$

(c) $\frac{1}{2}(1 + e_1e_2)e_3(1 - e_1e_2)$

(d) $\sqrt{1 + e_1e_2}$ (Hint: we can write this in exponential form, see next question, or think of it as a complex number $1 + j$.)

(e) Given a multivector $1 + \mathbf{v}$, find $(1 + \mathbf{v})^2$. Also, if $\mathbf{v}$ is a unit vector then what does $(1 + \mathbf{v})^2$ reduce to, hence find $(1 + \mathbf{v})^n$.

(f) Show that: $\frac{1}{2}(1 + e_2e_3)(e_1 + e_3)(1 - e_2e_3) = e_1 + e_2$.

Rotations in three dimensions

We can produce three bivectors in three dimensions, e_1e_2,e_3e_1 and e_2e_3, that are used as rotation operators. These replace quaternions or matrices.

(a) Confirm that they each square to minus one and anticommute with each other.

(b) Using the formula for the exponential $\mathrm{e}^x = 1 + x + x^2/2! + x^3/3! + \ldots$, show that $\mathrm{e}^{e_{12}\theta} = \cos\theta + e_{12}\sin\theta$, essentially deMoivres theorem. We use the shorthand $e_{12} = e_1e_2$.

(b) In the plane $B_1 = e_1e_2$, should you premultiply or postmultiply by B_1 to create a anti-clockwise rotation.(Hint see what effect the operator has on the vectors e_1 and e_2.)

(c) What effect does the operator B_1 have on the vector e_3 direction, that is, calculate B_1e_3.

(d) Correct the vector rotation in part (c) above using Hamiltons formula (see below) to rotate the vector $\mathbf{v} = e_1 + 2e_3$, by $\pi/2$ radians in the e_1e_2 plane. (Hint: You should start with $\mathrm{e}^{e_{12}\pi/4}(e_1 + 2e_3)\mathrm{e}^{-e_{12}\pi/4}$).

(e) Rotate the vector $\mathbf{v}$ by $\pi/2$ in the e_3e_1 plane.

(f) Rotate the vector $\mathbf{w} = e_1$ by π radians in the plane e_{12} using Hamiltons formula. What meaning does this give to the expression $\mathrm{e}^{e_{12}\pi} = -1$.

Note: Hamilton found that to correctly rotate vectors we need to form the bilinear transformation

$$\mathbf{v}' = \mathrm{e}^{\vec{B}\theta/2}\mathbf{v}\mathrm{e}^{-\vec{B}\theta/2} \tag{2}$$

where $\vec{B} = b_1e_2e_3 + b_2e_3e_1 + b_3e_1e_2 = i\vec{b}$ is a unit bivector, that is where $\vec{B}^2 = -1$.

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