# Geometric Algebra: A natural representation of three-space

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## Abstract

Historically, there have been many attempts to produce the appropriate mathematical formalism for modeling the nature of physical space, such as Euclid's geometry, Descartes' system of Cartesian coordinates, the Argand plane, Hamilton's quaternions, and finally Gibbs' vector system using the dot and cross products. We conclude however, that Clifford's geometric algebra (GA), provides the most elegant description of space. Supporting this conclusion, we firstly show how geometric algebra encapsulates the key elements of the competing formalisms, such as complex numbers, quaternions and the dot and vector cross products and secondly we show how it provides an intuitive representation and manipulation of points, lines, areas and volumes. We also provide two key examples where GA has been found to provide an improved description of physical phenomena, electromagnetism and quantum theory.

# I. INTRODUCTION

Einstein once stated, 'Everything should be made as simple as possible, but not one bit simpler', and in this paper we ask the question: 'What is the simplest mathematical representation of three-dimensional physical space that is nevertheless complex enough to satisfactorily describe all its key properties?'

The presence of five regular solids confirms the conclusion that we live in a three-dimensional world. If we lived in a world with four spatial dimensions, for example, we would be able to construct six regular solids, and in five dimensions and above we would find only three[1]. Also, the gravity and the electromagnetic force laws have been experimentally verified to follow an inverse square law to very high precision[2], indicating the absence of additional macroscopic dimensions beyond three space dimensions. Hence a three-dimensional coordinate system, as proposed by Descartes, appears to be a suitable overall framework. For three-space, however, as well as positional coordinates, we also need to be able to represent orientation or rotations at each point in this space. In the plane the algebra of complex numbers can be used for rotation and in three space, the algebra for rotations is given by Hamilton's quaternions. Hence, in order to form a unified algebra of three-space we need to integrate the complex numbers and quaternions within the framework of Cartesian coordinates. This was achieved by Clifford in 1873, who named his system, *Geometric Algebra* (GA).

## A. Historical development

Around 547 BCE, opposing the world view of his time, Thales expressed his belief that every event on the earth had natural rather than mythological causes[3]. Pythagoras extended this notion of natural causation, by postulating that numbers and their relationships, underly all things. This mathematical approach by Pythagoras was masterfully applied by Euclid to geometry, deriving his famous set of geometrical theorems based on a few simple axioms, that formed the first comprehensive theory for the physical world. The next real breakthrough in mathematical science did not come though till the seventeenth century, and it has been extensively debated by historians, why there was such a slow down in the progress of science and mathematics following the Greek explosion. Various suggestions have been provided to answer this, such as the Roman empire suppressing dissent and not sponsoring the arts, the ready availability of slaves obviating the need for work efficiencies[4]. However it has also been proposed that the algebraic and numerical system used by the Greeks, had inherent limitations, which were roadblocks to further progress[5]. For example the length along the diagonal of a unit square, we know today as $\sqrt{2}$, being an irrational number, did not exist in the Greek numeric system, which was based solely on integers and their ratios. Another hindrance would have been the roman numerals which made numeric manipulation difficult.

However with the arrival of Hindu-Arabic numbers in about 1000 AD into Europe, which included a zero that allowed positional representation for numbers, together with the acceptance of negative numbers in 1545 AD, allowed for the concept of a complete number line to be developed. This then paved the way for Descartes to revolutionize the Greek system in 1637, by proposing a union of algebra and geometry using Cartesian coordinates. He stated 'Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division and the extraction of roots, which may be considered a kind of division, so in geometry, to find required lines it is merely necessary to add or subtract lines.' Descartes thus postulated an equivalence between line segments and numbers, something the Greeks were not prepared to do. This achievement is identified by John Stuart Mill, 'the greatest single step ever made in the exact sciences'[6].

The Cartesian coordinate system proposed by Descartes, appears to become confused, however, with the later development of the Argand diagram, which, while isomorphic to the Cartesian plane, consists of one real and one imaginary axis, and so not rotationally symmetric. To add to the confusion, Hamilton in 1843 generalized the complex numbers to three space, defining the algebra of the quaternions using the basis elements $\mathrm{i}, \mathrm{j}, \mathrm{k}$ that can also be used as a substitute for three dimensional Cartesian coordinates. This confused state of affairs, on exactly how to represent three-space coordinates and rotations, was finally resolved by William Clifford in 1873. Clifford adopted the Cartesian coordinate system of Descartes, but then also integrated the algebra of complex numbers and quaternions as the rotation operators within this space. Clifford also achieved a fulfillment of Descartes' original vision of a vector being able to be manipulated in the same way as normal numbers, by deriving a multiplication and division operation for vectors that allowed them to be treated as algebraic variables. Additionally Clifford's system extended this idea to its natural conclusion, allowing allowing not just lines, but also areas and volumes, and their compositions, to be also treated in this same way.

### B. Clifford's definition of three-space

How did Clifford solve the problem of forming an integrated description of three-space combining Cartesian coordinates and the algebra of complex numbers and quaternions, as well as providing an algebraic treatment of lines, areas and volumes?

Firstly, we represent the three degrees of freedom in a Cartesian coordinate system by the algebraic constants $e_1, e_2$ and $e_3$ as shown in Fig. 1, which we define to have a positive square, that is $e_1^2 = e_2^2 = e_3^2 = 1$. The next crucial step is specify these elements as anticommuting, that is $e_j e_k = -e_k e_j$ for $j \neq k$. These few definitions are sufficient to define Clifford's system.

Geometrically, the basis elements $e_1, e_2, e_3$, the bivectors $e_2e_3$, $e_1e_3$ and $e_2e_3$ and the trivector $e_1e_2e_3$, represent unit lines, unit areas, and unit volumes respectively. We also find that the compound algebraic elements, the bivectors $e_2e_3$, $e_1e_3$ and $e_2e_3$ all square to minus one, for example, $(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1$, using the anticommutivity and positive square of the basis elements. We can now identify an isomorphism of the three bivectors with the three quaternions of Hamilton, so that $\mathrm{i} \leftrightarrow \mathrm{e_2e_3}$, $\mathrm{j} \leftrightarrow \mathrm{e_1e_3}$, $\mathrm{k} \leftrightarrow \mathrm{e_1e_2}$. Also, in the plane, the bivector $e_1e_2$ can be used as a replacement for the unit imaginary $\sqrt{-1}$, forming a complex-like number $a+ib$, where we define $i = e_1e_2$. The final compound element, the trivector $j = e_1e_2e_3$ also squares to minus one and commutes with all basis elements and so is isomorphic to the scalar unit imaginary $\sqrt{-1}$ in three dimensions. Now, because the unit imaginary is no longer required in Clifford's system, and because the unit imaginary was first used in complex numbers that are isomorphic to GA in two dimensions, we will adopt the widely used symbol $i = e_1e_2$ to represent the bivector, and in three dimensions, we will adopt $j = e_1e_2e_3$, a commonly used symbol in electrical engineering to represent the unit imaginary. This distinction between two forms of the unit imaginary $\sqrt{-1}$, as $i$ and $j$ in two and three dimensions respectively, has physical significance when describing Dirac's equation for the electron, described later. The basic elements of the algebra used to describe three dimensional space shown in Fig. 1.

Now, using the trivector $j$ we also find the relations $e_1e_2 = je_3$, $e_3e_1 = je_2$ and $e_2e_3 = je_1$. For example, $je_1 = e_1e_2e_3e_1 = e_1^2e_2e_3 = e_2e_3$, as required. These relations can be summarized by the relation $e_ie_j = j\epsilon_{ijk}e_k$, which we see describes the Pauli algebra, and hence we can use

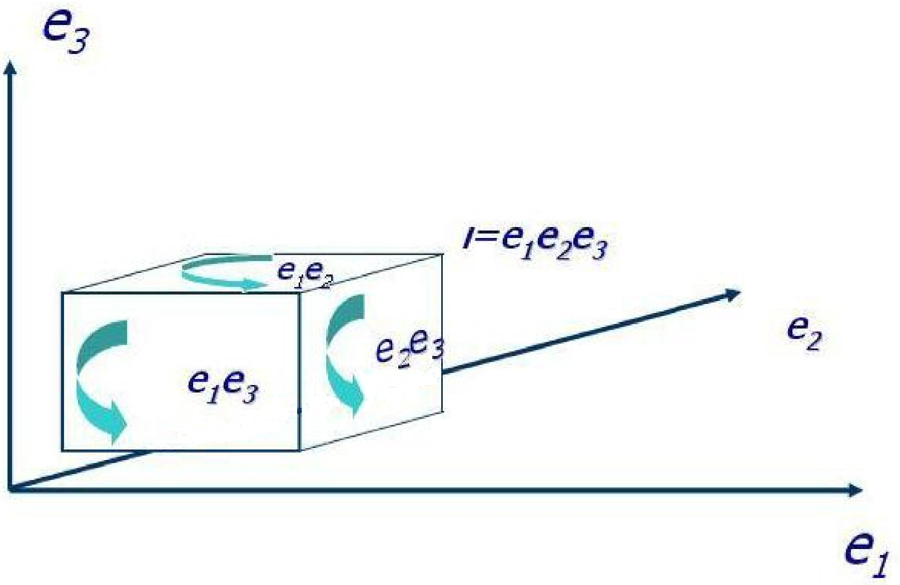


FIG. 1: The basic elements of Clifford's model for three space. This consists of three unit vectors $e_1, e_2$ and $e_3$, three unit areas $e_2e_3, e_3e_1$ and $e_1e_2$ and a unit volume $j = e_1e_2e_3$

Clifford's basis vectors $e_k$ to replace the three Pauli matrices $\sigma_k$ commonly used to describe quantum mechanical spin.

The competing mathematical systems that Clifford unified is shown in Fig. 2. In the plane the unit imaginary $\sqrt{-1}$ is replaced with the bivector $i = e_{12}$, using the subscript notation $e_{12} = e_1e_2$. Hamilton's three quaternions $\mathrm{i}, \mathrm{j}$ and $\mathrm{k}$ representing rotations about the three available axes, can be replaced with the bivectors $e_{23}$, $e_{13}$ and $e_{12}$ respectively, as shown, with the Cartesian axes described by unit vectors $e_1$, $e_2$ and $e_3$.

### C. The Clifford vector product

Using the three basis elements we can define a vector $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3$, where $v_i \in \Re$, and given a second vector $\mathbf{u} = u_1e_1 + u_2e_2 + u_3e_3$, we can find their algebraic product using

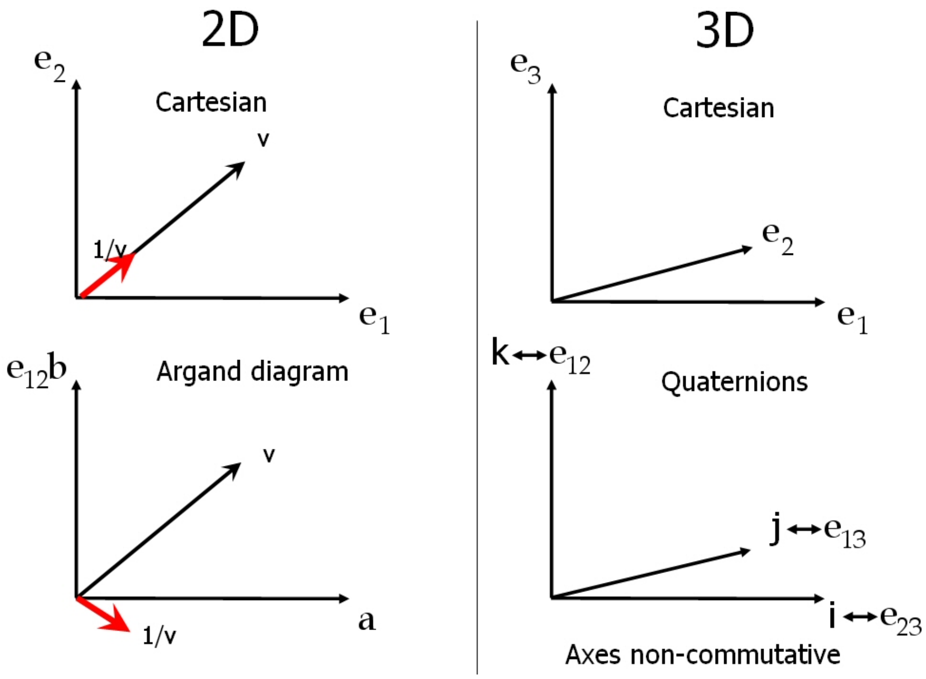


FIG. 2: Clifford's representation of three space. Firstly, the Cartesian plane is defined by the vectors $e_1$ and $e_2$, with the bivector $e_{12}$ identified as the unit imaginary used to define the Argand diagram. The inverse of a vector $\mathbf{v}$ is also shown in red, both for the Cartesian case and the for the Argand plane. In 3D, defined by the vectors $e_1, e_2$ and $e_3$, the non-commuting quaternions $\mathrm{i}, \mathrm{j}$ and k are replaced by the three bivectors $e_{23}$, $e_{13}$ and $e_{12}$ as shown.

the distributive law of multiplication over addition, giving

$$\begin{aligned}
&\mathbf{uv} \\
&= (e_1u_1 + e_2u_2 + e_3u_3)(e_1v_1 + e_2v_2 + e_3v_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3 + (u_2v_3 - v_2u_3)e_2e_3 + (u_1v_3 - u_3v_1)e_1e_3 + (u_1v_2 - v_1u_2)e_1e_2 \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v},
\end{aligned} \tag{1}$$

which produces a sum of the dot and wedge products. This algebraic product is commonly referred to as the geometric product. If we now use the dual relation $e_ie_j = j\epsilon_{ijk}e_k$ we can transform this result to

$$\begin{aligned}
\mathbf{uv} &= u_1v_1 + u_2v_2 + u_3v_3 + j\left((u_2v_3 - v_2u_3)e_1 + (u_1v_3 - u_3v_1)e_2 + (u_1v_2 - v_1u_2)e_3\right) \\
&= \mathbf{u} \cdot \mathbf{v} + j\mathbf{u} \times \mathbf{v},
\end{aligned} \tag{2}$$

which now forms a resultant in the form of a complex-like number consisting of the dot and cross products. Both the wedge product form and the cross product form are useful, though the dual relation, allowing the cross product form, only applies in three dimensions. Hence we can see that the the dot and the cross products indeed appear intrinsic to three dimensional space, however the advantage of the Clifford system is that they are unified into a single invertible number, as shown in Eq. (2). We can also identify a limitation of defining the cross product as a separately defined product, as it does not naturally extend to higher dimensional spaces, whereas the formulation in Eq. (1) does. The expression in Eq. (2) generated by simply expanding the brackets defining two vectors thus provides an alternative calculation tool to the conventional method of using the determinant of two vectors embedded in a $3 \times 3$ matrix. Clearly, following Clifford's treatment of vectors, as in Eq. (1), it is not possible treat vectors only in terms of traditional row or column vectors, but rather as a linear combination of the basis elements $e_1$ $e_2$ and $e_3$.

As can be seen from Eq. (1), for the case of a vector multiplied by itself, the wedge product will be zero and hence the square of a vector $\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v}$, becomes a scalar quantity. However this now allows us to define the inverse of a vector $\mathbf{v}$ as

$$\mathbf{v}^{-1} = \frac{1}{\mathbf{v}^2}\mathbf{v}. \tag{3}$$

This gives the result $\mathbf{v}\mathbf{v}^{-1} = \frac{1}{\mathbf{v}^2}\mathbf{v}\mathbf{v} = 1$ as required, so that we form the vector division $\frac{\mathbf{u}}{\mathbf{v}} = \mathbf{u}\mathbf{v}^{-1}$.

We can now compare the inverse of a Cartesian vector with the inverse of complex number. Given a complex number $z = r\mathrm{e}^{i\theta}$ we find the inverse $z^{-1} = (1/r)\mathrm{e}^{-i\theta}$, that has an inverse length, with a negative angle. For a Cartesian vector $\mathbf{v} = e_1 r\mathrm{e}^{i\theta} = r\cos\theta e_1 + r\sin\theta e_2$ in Clifford's system, we find the inverse vector $\mathbf{v}^{-1} = (1/r)e_1\mathrm{e}^{i\theta} = (1/r)\cos\theta e_1 + (1/r)\sin\theta e_2$ that is a vector of inverse length but in the same direction as the original vector, as shown in Fig. 2. The negative direction for the angle $\theta$ for the case of the inverse of a complex number, forming an inverse rotation, also confirms their role as rotation operators rather than as a replacement for Cartesian vectors.

As a simple application of Clifford's geometric product, if we have two vectors $\mathbf{a} = a_1e_1 + a_2e_2 + a_3e_3$ and $\mathbf{b} = b_1e_1 + b_2e_2 + b_3e_3$, we can find a third vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$, as shown in Fig. 3. We then find that

$$\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a} = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b} \tag{4}$$

using the the result from Eq. (2) that $\mathbf{ab} + \mathbf{ba} = \mathbf{a} \cdot \mathbf{b} + j\mathbf{a} \times \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + j\mathbf{b} \times \mathbf{a} = 2\mathbf{a} \cdot \mathbf{b} = 2|\mathbf{a}||\mathbf{b} \cos C$, where $C$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, a result also known as the cos rule for triangles.

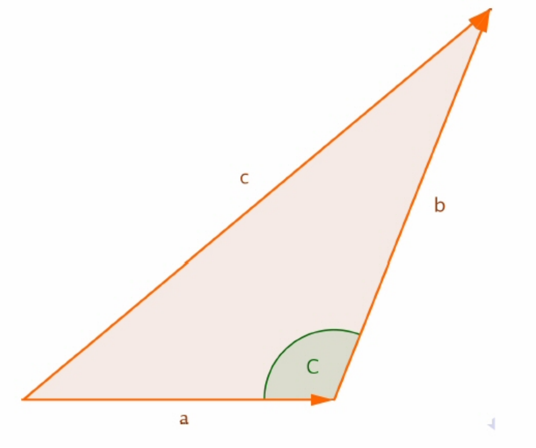


FIG. 3: Deriving the Cos rule for triangles. From the diagram we have the vector relation that $\mathbf{c} = \mathbf{a} + \mathbf{b}$, that gives using the geometric product $\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b}$.

### D. Example 1: Area calculation

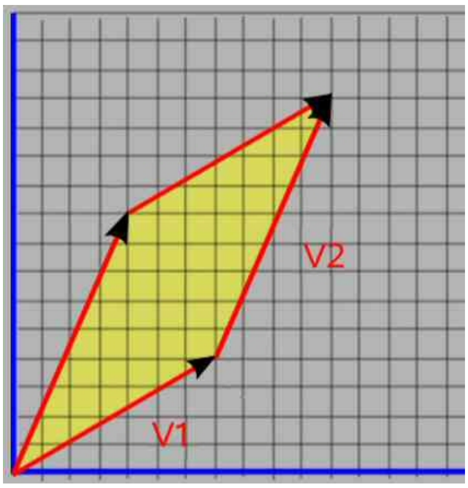


FIG. 4: Calculating areas using the geometric product.

Inspecting Fig. 4, we might wish to know the area enclosed by the two vectors, which we can calculate from a variety of geometrical constructions, to be $u_1 v_2 - u_2 v_1$. Alternatively, from Eq. (1), we can write the product of the two vectors

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = u_1 v_1 + u_2 v_2 + (u_1 v_2 - u_2 v_1) e_1 e_2, \tag{5}$$

then we can see that the area is the bivector term $\mathbf{u} \wedge \mathbf{v}$. The bivector $e_1 e_2$ represents a unit area, and so it is natural to expect this component to represent the area. Therefore we can write for the area

$$A = \langle \mathbf{uv} \rangle_2, \tag{6}$$

where the notation $\langle \mathbf{uv} \rangle_2$ means to retain the second grade or bivector terms. Dimensionally this also makes sense, because we are looking for a result with dimensions of area or meter$^2$. This argument also applies to three dimensions, where the volume will therefore need to be grade 3, that is for three vectors we find the enclosed volume $V = \langle \mathbf{uvw} \rangle_3$ as expected. Thus a routine calculation of the geometric product, followed by the selection of the desired components dimensionally, allows the relevant information to be extracted.

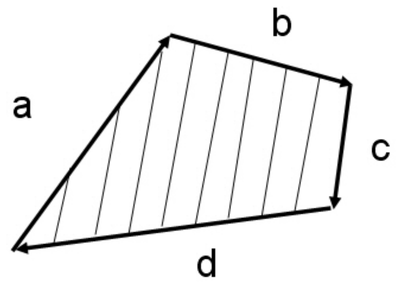


FIG. 5: Finding the area bounded by a set of vectors.

This principle can also be extended, and for a set of vectors that form a polygon, we can calculate the area as

$$A = \frac{1}{2} \langle \mathbf{ab} + \mathbf{bc} + \mathbf{cd} + \mathbf{da} \rangle_2 . \tag{7}$$

### E. The multivector

In GA, the basis elements $e_1, e_2$ and $e_3$ are algebraic constants and so we are free to combine the various scalar, vector, bivector and trivector components. In fact, adding all available components, we form the space of multivectors $\Re \oplus \Re^3 \oplus \bigwedge^2 \Re^3 \oplus \bigwedge^3 \Re^3$, an eight-dimensional real vector space also denoted by $Cl_{3,0}(\Re)$, which can be written

$$M = a + \mathbf{v} + j\mathbf{w} + jt, \tag{8}$$

| GA multivector | Alternate formalism | Description |
|---|---|---|
| $v_1e_1 + v_2e_2 + v_3e_3$ | $[v_1, v_2, v_3]^T$ | Vectors/Pauli matrices |
| $a + ib$, $i = e_{12}$ | $a + \mathrm{i}b$, $\mathrm{i} = \sqrt{-1}$ | Complex numbers |
| $a + be_{23} + ce_{13} + de_{12}$ | $a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ | Quaternions/Pauli spinors |
| $\mathbf{E} + j\mathbf{B}$ | $F^{\mu\nu}$, $\mu, \nu \in \{0, 1, 2, 3\}$ | Electromagnetic field tensor |
| $a + \mathbf{E} + j\mathbf{B} + jb$ | $\psi_\mu$ $\mu \in \{0, 1, 2, 3\}$ | Dirac electron wave function |

TABLE I: Comparison of GA with alternate mathematical formalisms.

which shows in sequence, a scalar $a$, vector $\mathbf{v} = v_k e_k$, bivector $j\mathbf{w} = jw_k e_k$ and trivector $jt$ terms, where $k \in \{1, 2, 3\}$. This general three-space multivector can be used to represent lines, areas and volumes within three dimensions but also a diverse range of physical phenomena. As already noted the bivectors are isomorphic to the three quaternions $\mathrm{i}, \mathrm{j}, \mathrm{k}$, and we find the multivector $a + j\mathbf{b} = a + jb_1e_1 + jb_2e_2 + jb_3e_3$ isomorphic to a quaternionic number $q = a + b_1\mathrm{i} - \mathrm{b}_2\mathrm{j} + \mathrm{b}_3\mathrm{k}$. We have already identified $e_1, e_2$ and $e_3$ with the Pauli matrices, and for a Pauli spinor, representing a spin-$\frac{1}{2}$ particle, we have the mapping

$$|\psi\rangle = \begin{bmatrix} a + \mathrm{i}a_3 \\ -a_2 + \mathrm{i}a_1 \end{bmatrix} \leftrightarrow \psi = a + ja_1e_1 + ja_2e_2 + ja_3e_3 = a + j\mathbf{a}, \tag{9}$$

where $\mathrm{i} = \sqrt{-1}$, also mapping to the even sub algebra of the multivector, which shows the equivalence of GA bivectors, Pauli spinors and quaternions[7].

We will also see later how the electromagnetic field antisymmetric tensor $F^{\mu\nu}$[8], maps as follows into the vector and bivector components of the multivector

$$F^{\mu\nu} \leftrightarrow F = \mathbf{E} + jc\mathbf{B}, \tag{10}$$

with the dual tensor $G^{\mu\nu}$ given in GA by $G = jF$. We have found that the scalars and bivectors can be used to represent the Pauli spinors, and the vector and bivector components used to describe the electromagnetic field and so we might ask if there is any physical phenomena that requires the full multivector, as shown in Eq. (8) for its representation. We find, in fact, that the wave function used to represent the electron in Dirac's relativistic wave equation, maps to the full multivector.

Hence the great versatility of the three-space multivector is demonstrated in Table I, being able to replace a large variety of mathematical structures and formalisms, as well as elegantly describe many physical phenomena[9].

### F. Common algebraic operations on a multivector

Descartes claimed that the five common algebraic operations of addition, subtraction, multiplication, division and square root, could be applied to his line segments, however this idea can now be extended to not only areas and volumes, but also to the full multivector shown in Eq. (8). The multivector represents a set of elements, containing not only a line, an areal element (bivectors) and a volume element (trivector). When algebraic operations are applied to these sets of geometric elements, we form a new set of geometric elements within the space of multivectors.

FIG. 6: A multivector $M = a + \mathbf{v} + j\mathbf{w} + jt$ representing a point, line, area and volume, that can be added, subtracted, multiplied or divided by other multivectors within the space of multivectors.

#### 1. *Working with multivectors*

Addition and subtraction are simply defined by adding like components, that is, if $M_1 = a_1 + \mathbf{v_1} + j\mathbf{w_1} + jt_1$ and $M_2 = a_2 + \mathbf{v_2} + j\mathbf{w_2} + jt_2$, then $M_1 + M_2 = (a_1 + a_2) + (\mathbf{v_1} + \mathbf{v_2}) + j(\mathbf{w_1} + \mathbf{w_2}) + j(t_1 + t_2)$ and similarly for subtraction.

The multiplication operation is given by an algebraic product, similar to the algebraic product of two vectors, that is

$$\begin{aligned}
M_1 M_2 &= (a_1 + \mathbf{v_1} + j\mathbf{w_1} + jt_1)(a_2 + \mathbf{v_2} + j\mathbf{w_2} + jt_2) \\
&= (a_1 a_2 + \mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{w}_1 \cdot \mathbf{w}_2 - t_1 t_2) + (a_2 \mathbf{v_1} + a_1 \mathbf{v_2} - t_2 \mathbf{w_1} - t_1 \mathbf{w_2} - v_1 \times w_2 + v_2 \times w_1) \\
&+ j(a_2 \mathbf{w_1} + a_1 \mathbf{w_2} + t_2 \mathbf{v_1} + t_1 \mathbf{v_2} + v_1 \times v_2 - w_1 \times w_2) + j\big(a_1 t_2 + a_2 t_1 + v_1 \cdot w_2 + w_1 \cdot v_2\big),
\end{aligned} \tag{11}$$

where we have used repeatedly the geometric product defined in Eq. (1), with the brackets showing the scalar, vector, bivector and trivector elements respectively.

For the general multivector $M = a+\mathbf{v}+j\mathbf{w}+jt$, it is useful to define two automorphisms. Firstly reversion, that reverses the order of the basis products, giving $\tilde{M} = a+\mathbf{v}-j\mathbf{w}-jt$ and space inversion $M^* = a-\mathbf{v}+j\mathbf{w}-jt$. We can then define $M^\dagger = \tilde{M}^* = a-\mathbf{v}-j\mathbf{w}+jt$, that gives $MM^\dagger = a^2-v^2+w^2-t^2+2j(at-\mathbf{v}\cdot\mathbf{w})$. We therefore find the inverse to $M$

$$M^{-1} = M^\dagger/(MM^\dagger). \tag{12}$$

The multivector inverse fails to exist when $MM^\dagger = 0$ or when $a^2+w^2 = v^2+t^2$ and $at = \mathbf{v}\cdot\mathbf{w}$, that we can write as the single condition $(\mathbf{v}+j\mathbf{w})^2 = (a+jt)^2$. The previously defined vector inverse in Eq. (3) now becomes simply a special case of the general multivector inverse. This formula also applies unchanged in a one or two dimensional space.

### 2. *The square root of a multivector*

In order to fully satisfy Descartes ideal of common algebraic operations being applicable to geometric quantities, such as lines and areas, we now finally seek the square root of a multivector. We show the simpler case of two dimensions, leaving the three dimensional case for the appendix.

Given a general two-dimensional multivector $M = a+\mathbf{v}+ib$, then seeking a multivector $N$, such that $N^2 = M$, we find

$$N = M^{\frac{1}{2}} = \pm\frac{1}{2c}\left(2c^2+\mathbf{v}+ib\right), \tag{13}$$

where we find from the quadratic formula $c^2 = \frac{a\pm\sqrt{a^2-\mathbf{v}^2+b^2}}{2}$. Now, because multivector multiplication is associative we can now find all the rational powers $M^{p/2^q}$, where $p,q$ are integers. The special cases where the square root fails to exist are $M = \mathbf{v}+i|\mathbf{v}|$, or a pure vector with $M = \mathbf{v}$. Also if $\mathbf{v}^2 > a^2+b^2$ then we will produce a complex number for $c^2$, but if we allow the solution space to expand to three dimensions this can be represented by the unit trivector $j$, giving $c^2 = \frac{a\pm j\sqrt{-a^2+\mathbf{v}^2-b^2}}{2}$. However to take powers of multivectors, such as square roots, it is more general to achieve this through logarithms and exponents.

**Exponential map of a multivector**

The exponential of a multivector is defined by constructing the Taylor series

$$\mathrm{e}^{M} = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots, \tag{14}$$

which is absolutely convergent for all multivectors $M$[5].

Given a three-dimensional multivector $a + \mathbf{v} + j\mathbf{w} + jt$, then defining $F = \mathbf{v} + j\mathbf{w}$, we find $F^2 = (\mathbf{v} + j\mathbf{w})^2 = \mathbf{v}^2 - \mathbf{w}^2 + 2j\mathbf{v}\cdot\mathbf{w}$. We then define $\sqrt{F^2} = \sqrt{-FF^\dagger} = j|F|$ and so $F = j\hat{F}|F|$, where $\hat{F} = F/\sqrt{F^2}$ and $\hat{F}^2 = 1$. Hence

$$\begin{aligned} \mathrm{e}^{a+\mathbf{v}+j\mathbf{w}+jt} &= \mathrm{e}^{a+jt}\mathrm{e}^{j|F|\hat{F}} \\ &= \mathrm{e}^{a+jt}\left(1 + j\hat{F}|F| - \frac{|F|^2}{2!} - \frac{j\hat{F}|F|^3}{3!} + \frac{|F|^4}{4!} + \ldots\right) \\ &= \mathrm{e}^{a+jt}\left(\cos|F| + j\hat{F}\sin|F|\right). \end{aligned} \tag{15}$$

If $|F| = 0$, then referring to the second line of the derivation above, we see that all terms following $j\hat{F}|F|$ are zero, and so, in this case $\mathrm{e}^{a+\mathbf{v}+j\mathbf{w}+jt} = \mathrm{e}^{a+jt}(1 + \mathbf{v} + j\mathbf{w})$.

We can thus write a multivector in polar form

$$a + \mathbf{v} + j\mathbf{w} + jt = r\mathrm{e}^{j\phi F/|F|} = r\left(\cos\phi + \frac{F}{|F|}\sin\phi\right), \tag{16}$$

where $r = |M|$ and $\phi = \operatorname{arccosh}\left(\frac{a+jt}{|M|}\right)$. Hence

$$\log(a + \mathbf{v} + j\mathbf{w} + jt) = \log|M| + \arccos\left(\frac{a+jt}{|M|}\right)\frac{F}{|F|}. \tag{17}$$

This result being a generalization of the well known result for quaternions, when $\mathbf{v} = t = 0$.

We can now also define the multivector power $M^P = \mathrm{e}^{\log(M)P}$, where $P$ can now also be generalized to a multivector.

For example, we could raise the multivector $2+\hat{v}$ to the power of the unit vector $\hat{v}$, giving

$$(2 + \hat{v})^{\hat{v}} = \mathrm{e}^{\log(2+\hat{v})\hat{v}} = 2 + \hat{v}, \tag{18}$$

using Eq. (17) and Eq. (15).

### G. The geometry of the multivector

Having shown how the multivector, representing a set of geometric elements, consisting of a point, line, area and volume, is subject to the common algebraic operations, we can now ask some elementary geometrical questions such as: What is the result of multiplying a line by an area? We can calculate this for a line $\mathbf{v}$ and a generally oriented area $j\mathbf{w}$ as $\mathbf{v}j\mathbf{w} = j\mathbf{v} \wedge \mathbf{w} + j\mathbf{v} \cdot \mathbf{w} = -\mathbf{v} \times \mathbf{w} + j\mathbf{v} \cdot \mathbf{w}$. As might have been expected this forms a volume $j\mathbf{v} \cdot \mathbf{w}$, and if the line is not perpendicular to the plane, we also produce a line, given by the vector $-\mathbf{v} \times \mathbf{w}$ in the plane of $j\mathbf{w}$ and orthogonal to $\mathbf{v}$, as shown in Fig. 7.

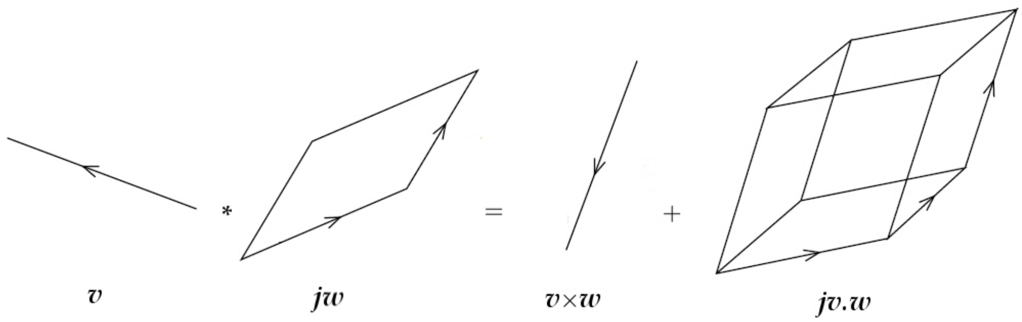


FIG. 7: A line $\mathbf{v}$ multiplied by an area $j\mathbf{w}$. This produces a volume $j\mathbf{v} \cdot \mathbf{w}$ as expected but also a line $-\mathbf{v} \times \mathbf{w}$ perpendicular to $\mathbf{v}$ in the plane of $j\mathbf{w}$.

### H. Reflection of vectors

Assuming a light ray with an incident vector $\mathbf{a}$, is impinging on a plane mirror $j\hat{n}$, with a unit normal $\hat{n}$, find the reflected vector. We find the reflected vector

$$\mathbf{b} = -\hat{n}\mathbf{a}\hat{n}. \tag{19}$$

If we reflect $\mathbf{b}$ in the same mirror we will recover the original vector $\mathbf{a}$, however if we reflected $\mathbf{b}$ in a slight rotated mirror plane $j\hat{m}$ we will in fact find a rotated vector

$$\mathbf{b} = \hat{m}\hat{n}\mathbf{a}\hat{n}\hat{m} = (\hat{m} \cdot \hat{n} + \hat{m} \wedge \hat{n})\mathbf{a}(\hat{n} \cdot \hat{m} + \hat{n} \wedge \hat{m}). \tag{20}$$

Now we have the unit bivector $\hat{B} = \hat{m} \wedge \hat{n}/\sin\theta$ describing the plane of rotation and $\theta = \arccos \hat{m} \cdot \hat{n}$ is the angle between the vectors $\hat{m}$ and $\hat{n}$, then

$$\mathbf{b} = (\cos\theta + \hat{B}\sin\theta)\mathbf{a}(\cos\theta - \hat{B}\sin\theta) = \mathrm{e}^{\theta\hat{B}}\mathbf{a}\mathrm{e}^{-\theta\hat{B}}, \tag{21}$$

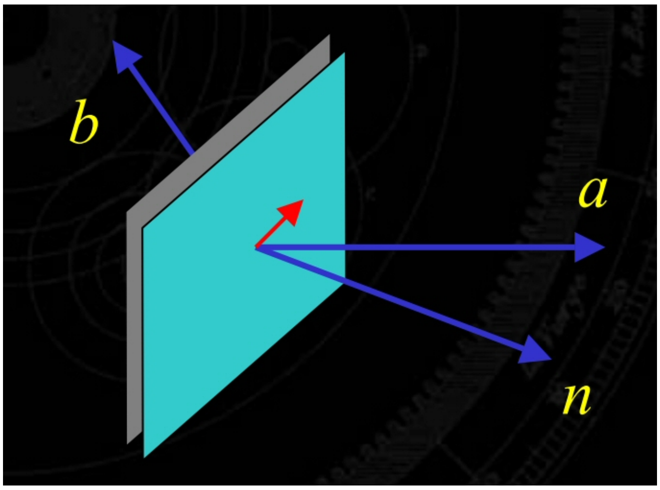


FIG. 8: An light ray incident on a plane mirror $j\hat{n}$. We find the reflected ray $\mathbf{b} = -\hat{n}\mathbf{a}\hat{n}$.

which will rotate the vector $\mathbf{a}$ an angle of $2\theta$ radians in the plane described by the unit bivector $\hat{B}$.

### I. Rotation of vectors

If we wish to rotate a vector $\mathbf{v}$ by an angle $\theta$, then we use the operation,

$$\mathbf{v}' = R\mathbf{v}\tilde{R} = \mathrm{e}^{j\mathbf{w}\theta/2}\mathbf{v}\mathrm{e}^{-j\mathbf{w}\theta/2} \tag{22}$$

where $R = \mathrm{e}^{j\hat{w}\theta/2}$. The unit bivector $j\hat{w}$ sets the plane of rotation, with a perpendicular axis $\hat{w}$, that rotates all vectors $\theta$ radians within this plane. In two dimensions this formula reduces to the single sided operator $\mathbf{v}' = \mathrm{e}^{i\mathbf{w}\theta}\mathbf{v}$ due to the anticommuting nature of $i = e_{12}$ over vectors. The rotation formula in two dimension now analogous to the conventional formula for the rotation of vectors in the Argand plane.

Rotations in geometric algebra are superior to orthogonal matrices in representing 3D rotations on four points: (i) It is easier to determine the bivector representation of a rotation than the matrix representation, (ii) they avoid the problem of gimbal lock, (iii) It is more efficient to multiply bivectors than matrices, and (iv) if a bivector product is not quite normalized due to rounding errors, then we simply divide by its norm, whereas if a product of orthogonal matrices is not orthogonal, then we need to use Gram-Schmidt orthonormalization, which is numerically expensive and not canonical.

### J. Precession of a spin-$\frac{1}{2}$ particle

We can model a mixed state quantum spin-$\frac{1}{2}$ particle by the multivector

$$\rho = \frac{1}{2}(1 + \mathbf{u}) \tag{23}$$

where the unit vector $\mathbf{u}$, represents the polarization axis of the particle. This approach is in fact equivalent to the density matrix formulation of quantum mechanics. We have $\rho^2 = \rho$ when $\mathbf{u}$ a unit vector, and if $\mathbf{u}$, is less than a unit vector, then we are modeling a mixed state. So placing our particle in a magnetic field $\mathbf{B}$, then using the rotation formula in Eq. (22), we find the precession about the $\mathbf{B}$ direction, given by

$$\rho' = \mathrm{e}^{j\mathbf{B}t}\rho\mathrm{e}^{-j\mathbf{B}t} = \frac{1}{2}(1 + \mathrm{e}^{j\mathbf{B}t}\mathbf{u}\mathrm{e}^{-j\mathbf{B}t}), \tag{24}$$

where we can see the precession is steady in time, and the rate of precession given by the strength of the field.

### K. Interpreting solutions of quadratics using GA

Imaginary numbers first appeared as the roots to quadratic equations, such as $x^2+4=0$, but Gauss noted in 1825 that 'The true metaphysics of the square root of minus one is elusive'.

However, with GA we can now supply a real geometrical solution to this equation, using the unit area, as $x = 2i$, recalling that $i = e_{12}$, that on substitution gives $(2e_{12})^2 + 4 = 0$, that indeed solves the equation. In fact many geometrical square roots of minus one exist, and in two dimensions we can write a general solution to $M^2 = -1$, as $M = \mathbf{x} + i\sqrt{1+\mathbf{x}^2}$, where $\mathbf{x} = x_1 e_1 + x_2 e_2$.

Hence in GA, we can write a solution $x = a + ib$, that from deMoiver's theorem, gives $R = r\mathrm{e}^{i\theta} = r(\cos\theta + i\sin\theta)$. In two-space we can rotate vectors using the equation

$$\mathbf{v}' = R\mathbf{v}, \tag{25}$$

that will rotate a vector $\mathbf{v} = v_1 e_1 + v_2 e_2$ by $\theta$ radians in a clockwise direction. For example, if $R = \mathrm{e}^{i\pi/2} = i = e_{12}$, then if $\mathbf{v} = e_2$, then $\mathbf{v}' = R\mathbf{v} = e_1 e_2 e_2 = e_1$, or a clockwise rotation by ninety degrees. Also, if we seek to rotate a vector by $\pi$ radians then we find

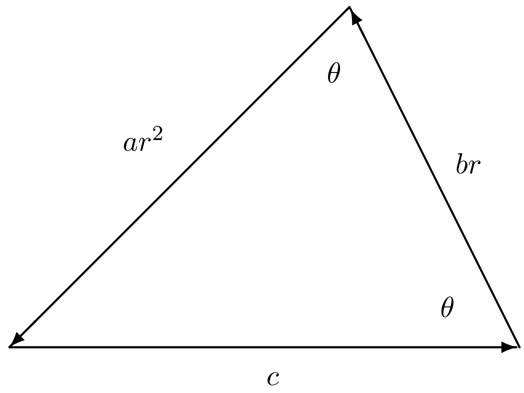


FIG. 9: Graphical solution to a quadratic equation. To solve graphically, we need to vary $r \in \Re$ and $\theta \in [0, \pi/2)$ while ensuring the arrows close in a triangle. Note that we have an Isocelles triangle and real solutions correspond to $\theta = 0$.

that $\mathbf{v}' = \mathrm{e}^{i\pi}\mathbf{v} = -\mathbf{v}$. This also illuminates the mysterious formula $\mathrm{e}^{i\pi} = -1$, that simply means in this context, that rotating a vector by $\pi$ radians flips its sign.

Hence solutions of quadratics using complex numbers, imply we are using rotation operators in the plane, instead of simply scaling along the real number line. This also explains why we always have two symmetrical complex solutions, if they exist, as they represent $\pm\theta$ directions for the rotation operation. So given a quadratic equation $ax^2 + bx + c = 0$, and substituting a rotor solution $x = -r\mathrm{e}^{-i\theta}$, we can acting with the quadratic on a general vector $\mathbf{v}$ on the left, to produce the vector equation

$$ar^2\mathrm{e}^{2i\theta}\mathbf{v} - br\mathrm{e}^{i\theta}\mathbf{v} + c\mathbf{v} = 0 \tag{26}$$

where we used the property of exponentials that $(\mathrm{e}^{i\theta})^2 = \mathrm{e}^{2i\theta}$. Hence, in order to solve the quadratic these three vectors must sum to zero, that can be shown visually in Fig. 9, where, without loss of generality, we have chosen a reference direction $\mathbf{v} = e_1$. From Fig. 9 we find $r^2 = \frac{c}{a}$ that gives $r = \sqrt{\frac{c}{a}}$ and $\theta = \arccos\left(\frac{b}{2\sqrt{ac}}\right)$.

### 1. *Quadratic equation example*

Assuming we are required to solve the quadratic

$$x^2 + x + 1 = 0 \tag{27}$$

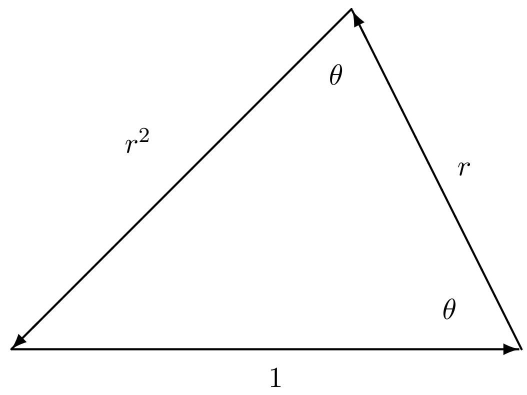


FIG. 10: Solving $x^2 + x + 1 = 0$ graphically. From the Isocelles triangle we note that $r = \pm 1$, and hence we have an equilateral triangle, which implies $\theta = \pi/3$. Therefore $x = -r\mathrm{e}^{\pm i\theta} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

then we find the Isocelles triangle shown in Fig. 10. From the property of Isocelles triangles we have $r = 1$ and we therefore realise, in this case, that we have an equilateral triangle, and hence $\theta = \frac{\pi}{3}$, and hence we have the solution

$$x = -\mathrm{e}^{\pm i\pi/3} = -\frac{1}{2} \pm e_{12}\frac{\sqrt{3}}{2} \leftrightarrow -\frac{1}{2} \pm \sqrt{-1}\frac{\sqrt{3}}{2}, \tag{28}$$

in agreement with the quadratic formula.

Other extensions now present themselves for the quadratic equation, such as expanding the solution space further to allow $x$ to be a quaternion (represented by bivectors), or to promote $x$, and $a, b, c$ to become full multivectors.

### L. Maxwell's equations in GA

Electromagnetism is one of the foundational theories of physics and Maxwell's equations were first published in 1865[10]. Maxwell's original equations were written for three-space, requiring 12 equations in 12 unknowns. These equations were later rewritten by Heaviside and Gibbs, in the newly developed formalism of dot and cross products, which reduced them to the four equations now seen in most modern textbooks[8] and shown below in S.I. units

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon}, \quad \text{(Gauss' law);} \\
\nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0, \quad \text{(Faraday's law);} \\
\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} &= \mu_0 \mathbf{J}, \quad \text{(Ampère's law);} \\
\nabla \cdot \mathbf{B} &= 0, \quad \text{(Gauss' law of magnetism),}
\end{aligned} \tag{29}$$

where $\mathbf{E}, \mathbf{B}, \mathbf{J}$ are conventional vector fields, with $\mathbf{E}$ the electric field strength and $\mathbf{B}$ the magnetic field strength and $\nabla = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z}$ the three gradient.

However inspecting the form of the geometric product for $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + j\mathbf{u} \times \mathbf{v}$, we can see that these equations can now be combined. If we multiply the second and fourth equations by $j$, then the first and second equations can be combined along with the third and fourth to give

$$\begin{aligned}
\nabla \mathbf{E} + \partial_t j\mathbf{B} &= \frac{\rho}{\epsilon} \\
\nabla j\mathbf{B} + \frac{1}{c^2}\partial_t \mathbf{E} &= -\mu_0 \mathbf{J},
\end{aligned} \tag{30}$$

where $\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + j\nabla \times \mathbf{E}$. However, these two remaining equations can now be added to produce

$$(\frac{1}{c}\partial_t + \nabla)(\mathbf{E} + jc\mathbf{B}) = \frac{\rho}{\epsilon} - c\mu_0 \mathbf{J}. \tag{31}$$

If we define the electromagnetic field $F = \mathbf{E} + jc\mathbf{B}$ and the four-gradient $\partial = \frac{1}{c}\partial_t + \nabla$, with the source $J = \frac{\rho}{\epsilon} - c\mu_0 \mathbf{J}$, we find

$$\partial F = J. \tag{32}$$

We can see that the $\mathbf{B}$ field, is written as a pseudovector $j\mathbf{B}$, as part of the field $F$. The different nature of the $\mathbf{E}$ and $\mathbf{B}$ fields is evident from the GA formalism but obscured in the tensor and Gibbs' vector formalism, where both are represented as polar vectors.

Also if we wished to describe Maxwell's original four equations as shown in Eq. (29) in plain English we would find it very cumbersome, however with GA, inspecting Eq. (32), we can simply say that the gradient of the field $F$ observed is proportional to the electromagnetic sources present.

### M. The Dirac equation

The Dirac equation is the relativistic wave equation describing spin-$\frac{1}{2}$ particles. We find using GA, that we can write the free Dirac equation in real three-space as

$$\partial F = -mF^* i \tag{33}$$

where the field $F$ is now the full multivector $F = a + \mathbf{E} + j\mathbf{B} + jb$.

The similarity of Dirac's equation with Maxwell's equation now becomes evident, comparing Eq. (33) and Eq. (32). Also, with the GA form of the Dirac equation, we can see that it describes a multivector field, given by Eq. (8), over real three-space, that is, at each point in three-space we have a multivector valued field defined. This is clearly a significantly simplified representation of the Dirac equation, which is normally considered embedded in four-dimensional spacetime employing $4 \times 4$ complex matrices, as shown in Appendix B.

## II. CONCLUSION

In this paper we ask the question 'What is the simplest and most natural mathematical representation of three-dimensional physical space?' and we conclude that GA provides the most natural formalism providing an intuitive representation of the geometric quantities points, lines, areas and volumes and subsuming the algebra of complex numbers and quaternions into a algebraic system over a real field. We claim that Clifford's GA is the simplest mathematical system conceivable that successfully represent the key properties of physical space. We illustrate the improved representation with the two examples of Maxwell's equations and the Dirac equation, both being written as single equations in real three dimensional space.

We also show how the multivector, shown in Eq. (8), can be viewed as a generalized number, useful in representing different physical and geometrical quantities, amenable to the basic operations of addition, subtraction, multiplication, division and square root. We showed how a general quadratic equation can be solved without recourse to complex numbers, giving the solutions geometric meaning as rotations in the plane.

We have adopted the symbols $i = e_{12}$ in two dimensions and $j = e_{123}$ for three dimensions as two geometric replacements for the generic scalar unit imaginary $\sqrt{-1}$, and we noted a distinction between $i$ and $j$ in the GA form of the Dirac equation in Eq. (33). We also think that this notation has merit when extended to higher dimensions. For four and five dimensions for example, we suggest the notation $I = e_{1234}$ and $J = e_{12345}$, where the capitalization indicates that they have a positive square, but capital $I$ like lower case $i$ is anticommuting and $J$ is commuting similar to $j$. As we continue to higher dimension this patterns of positive or negative squares, and commuting versus anticommmuting pseudoscalars repeats with a period of four.

The development of GA is now expanding rapidly, with benefits being found in research into black holes[11], quantum field theory[12], quantum tunneling[13], quantum computing[14], general relativity and cosmology[15], beam dynamics and buckling[16], computer vision[17] and EPR-Bell experiments[18].

Many commentators believe that Cliffords mathematical system 'should have gone on to dominate mathematical physics'[19], but, Clifford died young, at the age of just 33 and vector calculus was heavily promoted by Gibbs and rapidly became popular, eclipsing Clifford's work, which in comparison appeared strange with its non-commuting variables. In hindsight, non-commuting reflects the non-commutivity of rotations in three-space, and hence is exactly what is required for these variables. Gibb's system of vectors was fairly efficient with regard to Maxwell's equations, but with the new scientific discoveries of quantum mechanics and relativity if was found that standard vector analysis needed to be supplemented by many other mathematical techniques such as: tensors, spinors, matrix algebra, Hilbert spaces, differential forms etc. and as noted in[20], 'The result is a bewildering plethora of mathematical techniques which require much learning and teaching, which tend to fragment the subject and which embody wasteful overlaps and requirements of translation.' Conversely as we have seen GA is a natural formalism for not only Maxwell's equations (Eq. (32) and also quantum mechanics (Eq. (33) but also special relativity[21].

We also have shown that geometric algebra provides a natural representation of the basic properties of physical space, allowing intuitive manipulation of lines, areas and volumes using elementary algebraic operations, such as addition and multiplication. Vectors can now be treated like normal algebraic quantities that also have an inverse, with the added simplification that the dot and cross products do not need to be separately defined but are produced as a byproduct from the geometric product. Hence it would appear to be an excellent formalism to introduce into the school curriculum as a powerful tool for basic geometrical analysis of space, that also a formalism that can be extended to the study of university level subjects in electromagnetism, quantum theory and special relativity.

## III. APPENDIX

### A. Dirac equation

Dirac extended Schrödinger's and Pauli's equation into a relativistic setting in 1928, producing the equation

$$\gamma^\mu \partial_\mu \psi = -m\psi \mathrm{i}, \tag{34}$$

where $\mathrm{i} = \sqrt{-1}$ and which uses the Einstein summation convention, where

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{35}$$

The gamma matrices satisfy the relation and

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{36}$$

as expected for a set of orthonormal basis vectors. Hence, the opinion of many people, that Dirac rediscovered Clifford's geometric algebra with its anti-commuting basis vectors.

Dirac's complex, four-space equation using $4 \times 4$ complex matrices, is isomorphic to the real three-space version, shown in Eq. (33).

### B. The square root of a three dimensional multivector

To find the square root of a multivector $M$, we need to find a multivector $N$, such that $N^2 = M$ where

$$M = a + \mathbf{v} + j\mathbf{w} + jb \tag{37}$$

is a general multivector. In order to simplify calculations we write

$$M = (a + jb)(1 + \mathbf{m} + j\mathbf{n}) \tag{38}$$

where by inspection $\mathbf{m} = \frac{a\mathbf{v}+b\mathbf{w}}{a^2+b^2}$, $\mathbf{n} = \frac{a\mathbf{w}-b\mathbf{v}}{a^2+b^2}$ . Clearly this result is not valid if $a = b = 0$ implying a multivector $M = \mathbf{v} + j\mathbf{w}$, which can be found to have the solution $\sinh\theta =$

$-\frac{q_0}{p_0} \pm \sqrt{\frac{q_0^2}{p_0^2}+1}$, where $q_0 = \frac{v^2-w^2}{4}$ and $p_0 = \frac{1}{2}\mathbf{v}\cdot\mathbf{w}$. A subclass of solutions is also produced if $p_0 = 0$, giving $c = d = q_0^{1/4}$. The case with $q_0 = 0$ in fact appears to have no solution, that is the multivector $M = \mathbf{u} + j\mathbf{u}^\perp$, has no square root, although it can be approximated arbitrarily closely with $N = \epsilon + \frac{\mathbf{u}}{2\epsilon} + \frac{i\mathbf{u}^\perp}{2\epsilon} + \epsilon$, giving $N^2 \approx M = \mathbf{u} + j\mathbf{u}^\perp + 2j\epsilon^2$. So for cases with $a, b \neq 0$ we seek a multivector such that

$$\begin{aligned} &= (c + \mathbf{x} + j\mathbf{y} + jd)(c + \mathbf{x} + j\mathbf{y} + jd) \\ &= (c^2 + x^2 - y^2 - d^2) + 2(c\mathbf{x} - d\mathbf{y}) + 2j(c\mathbf{y} + d\mathbf{x}) + 2j(cd + \mathbf{x}\cdot\mathbf{y}) \\ &= 1 + \mathbf{m} + j\mathbf{n}. \end{aligned} \tag{39}$$

We can see that now we have constructed a simplified problem, of the square root of a multivector with $b = 0$, and $a = 1$, but from which we can reconstruct the full square root. Inspecting the six linear equations formed from the vector and trivector components we can see that we require

$$\mathbf{x} = \frac{d\mathbf{n} + c\mathbf{m}}{2h} \ , \ \ \mathbf{y} = \frac{c\mathbf{n} - d\mathbf{m}}{2h} \tag{40}$$

where $h = c^2 + d^2$. For the case $h = 0$, which implies $c = d = 0$, giving $N^2 = a + jb$, showing that $N$ is the square root of complex-like numbers. The square root of a complex number is calculated later in the appendix, and so for more general cases we can assume $h > 0$. We now need to just find $c$ and $d$ from the two remaining equations in Eq. (39) of

$$\begin{aligned} c^2 - d^2 + \frac{(m^2-n^2)(c^2-d^2)}{4h^2} + \frac{4cd\mathbf{m}\cdot\mathbf{n}}{4h^2} &= 1 \\ cd - \frac{cd(m^2-n^2) + (c^2-d^2)\mathbf{m}\cdot\mathbf{n}}{4h^2} &= 0 \end{aligned} \tag{41}$$

and after multiplying through by $h^2$ and making the replacement $q = \frac{m^2-n^2}{4}$ and $p = \frac{1}{2}\mathbf{m}\cdot\mathbf{n}$, we find

$$\begin{aligned} (c^2-d^2)(c^2+d^2)^2 + q(c^2-d^2) + 2cdp &= (c^2+d^2)^2 \\ 2cd(c^2+d^2)^2 - 2cdq + (c^2-d^2)p &= 0. \end{aligned} \tag{42}$$

We have difficult simultaneous polynomials in $c$ and $d$, and so it is now convenient to make the substitution

$$c = r\cosh\frac{\theta}{2} \ , \ \ d = r\sinh\frac{\theta}{2}, \tag{43}$$

producing

$$\begin{aligned}
r^4\cosh^2\theta + q + p\sinh\theta &= r^2\cosh^2\theta \qquad (44)\\
r^4\sinh\theta\cosh^2\theta - \sinh\theta q + p &= 0.
\end{aligned}$$

From the second equation in Eq. (44), we find

$$r = \pm\left(\frac{q - p\,\mathrm{cosech}\theta}{\cosh^2\theta}\right)^{\frac{1}{4}}. \qquad (45)$$

We can see that either sign for $r$ satisfies the two equations of Eq. (44) although this will only flip the overall sign of the square root, as expected, giving us a square root for the multivector of either sign. The term under the fourth root, $q - p\,\mathrm{cosech}\theta$ in fact remains non-negative, and hence $r$ remains real. On substitution back into the first equation in Eq. (44) we find the trigonometric equation,

$$2q - p\,\mathrm{cosech}\theta(1 - \sinh^2\theta) = \cosh\theta\sqrt{q - p\,\mathrm{cosech}\theta} \qquad (46)$$

and substituting $x = \sinh\theta$ we find the quartic equation in $x$

$$(q-p^2)x^4 - p(1+4q)x^3 + \left(2p^2 - q(4q-1)\right)x^2 + p(4q-1)x - p^2 = 0 \qquad (47)$$

which has a solution

$$x = \frac{p(1+4q+s) \pm u}{4(q-p^2)}, \qquad (48)$$

where $s = +\sqrt{(4q-1)^2 + 16p^2}$ and $u = \sqrt{2}\sqrt{16q^3 + 12qp^2 + p^2 - 4q^2 + s(p^2 + 4q^2)}$. We can see that $s$ remains real, but if we allow a negative square root of $s$ inside the equation for $u$, then we will produce an imaginary term and hence we need to maintain a positive square root for $s$ which leaves just the two real solutions to the quartic as shown in Eq. (48). Hence we have two distinct $\theta = \mathrm{arcsinh}x$, which imply two distinct square roots, ignoring signs. We can see that there is a special case $q = p^2$ that will reduce Eq. (47) to a cubic,

$$p(1+4p^2)x^3 + p^2(4p^2-3)x^2 - p(4p^2-1)x + p^2 = 0, \qquad (49)$$

which has the single real solution $x = -p$, giving a single root with $\theta = -\mathrm{arcsinh}p$. If $p = 0$, then the cubic fails and we find two cases dependent on $q$, with firstly $\cosh\theta = 2\sqrt{q}$, requiring $q \geq \frac{1}{4}$ and secondly $\theta = 0$ with $r = c$, where $c = \sqrt{\frac{1+\sqrt{1-4q}}{2}}$, valid for $q \leq \frac{1}{4}$.

The square root of $a+jb$ is easily found to be $f+jg=\pm\frac{1}{\sqrt{2}}(\sqrt{a+\sqrt{h_0}}+j\sqrt{-a+\sqrt{h_0}})$ Hence for $M$ given by Eq. (37), we find the square root

$$M^{\frac{1}{2}}=\pm(c+\mathbf{x}+j\mathbf{y}+jd)\,(f+jg)\,, \tag{50}$$

where $c,d$ given by Eq. (43) and $\mathbf{x},\mathbf{y}$ given by Eq. (40).

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