Clifford's geometric algebra

A unifying mathematical formalism for science

by: Dr. James Chappell

". . . it is a good thing to have two ways of looking at a subject, and to admit that there are two ways of looking at it."

James Clerk Maxwell

What is geometric algebra?

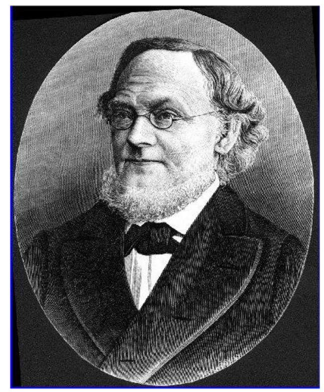
- An elegant mathematical framework for expressing geometrical ideas and doing computations.
- used in physics, engineering and in computer vision applications.

Its main advantages are:

- geometrical ideas can be expressed compactly without having to consider coordinates and bases.
- Correctly expresses the properties of physical space
- Can be extended effortlessly to spaces of arbitrary dimensions.
- Unifies complex numbers, quaternions, vectors,Pauli spinors….
- Resolves distinction between true and axial vectors.

Pioneers of geometric algebra

Hamilton
1843



Grassman

Clifford
1879

Quaternions
i,j,k

Wedge product

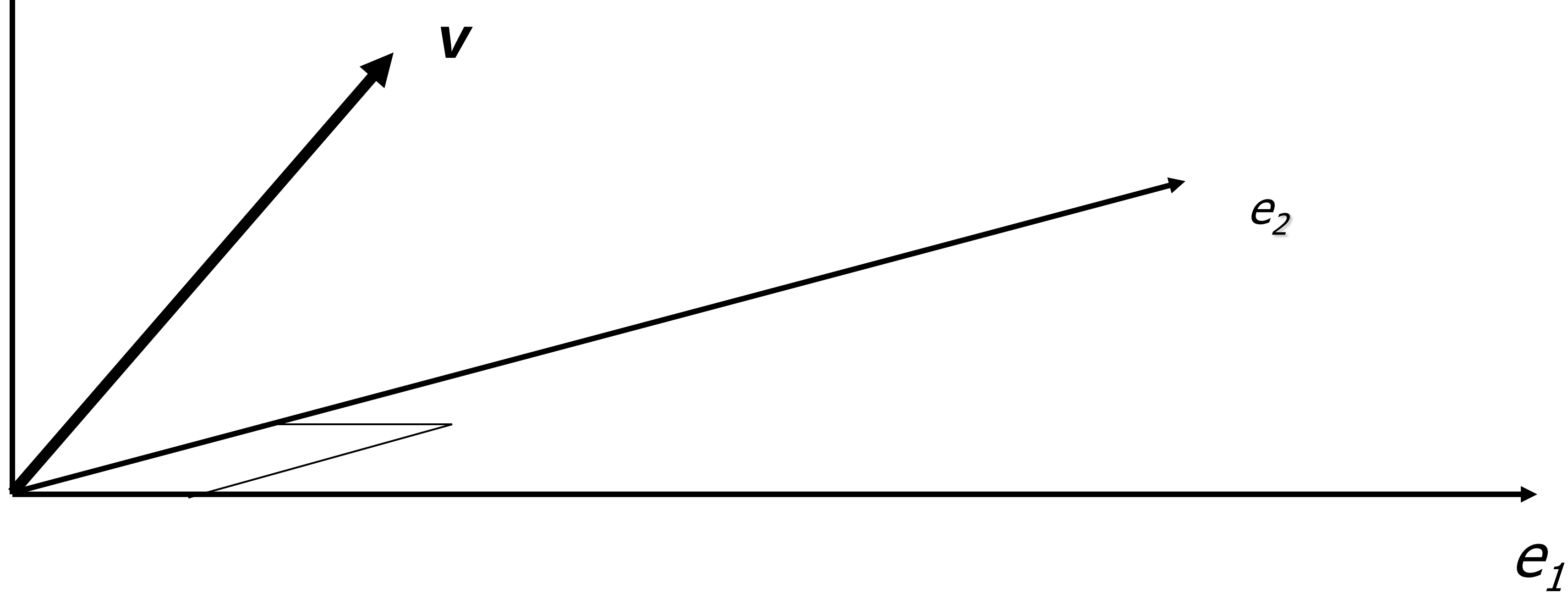
Geometric product
Unifies dot and cross product

$$\boldsymbol{ab} = \boldsymbol{a}.\boldsymbol{b} + i\,\boldsymbol{a} \times \boldsymbol{b}$$

Defining vectors (Cartesian coordinates)

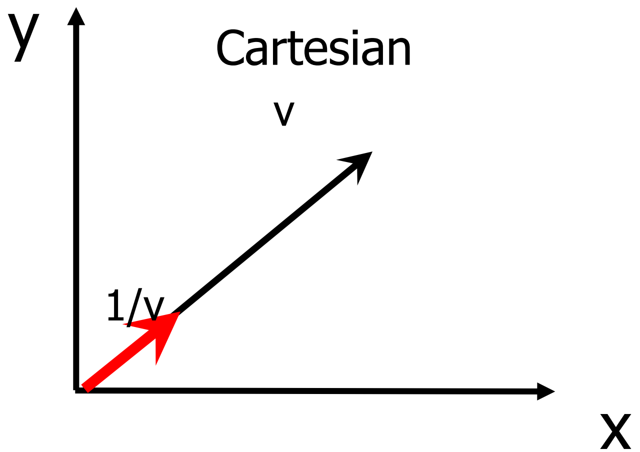
Descartes 1637

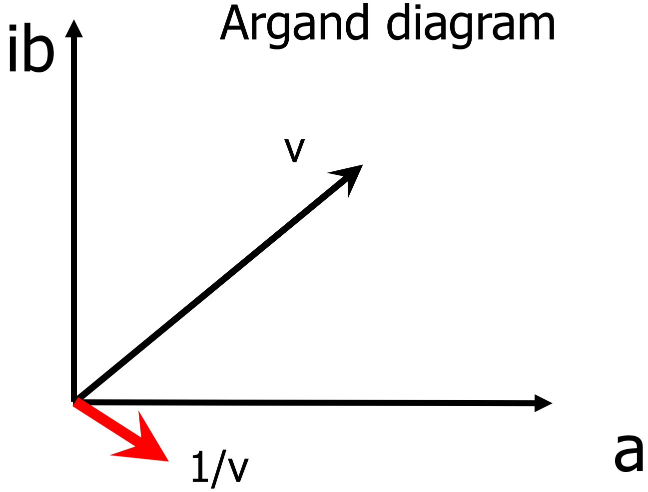
- Define three orthogonal axes e_1, e_2, e_3
- Produce a vector $\boldsymbol{v} = a_1 e_1 + a_2 e_2 + a_3 e_3$



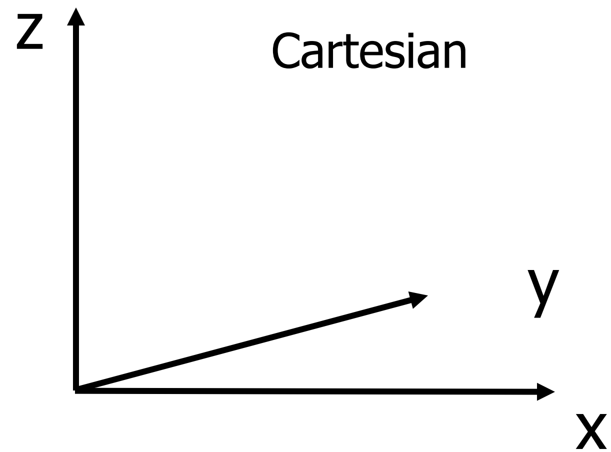
Rival coordinate systems

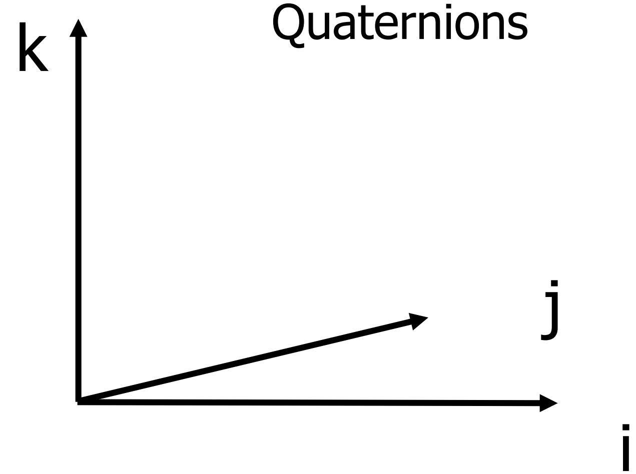
2D





3D



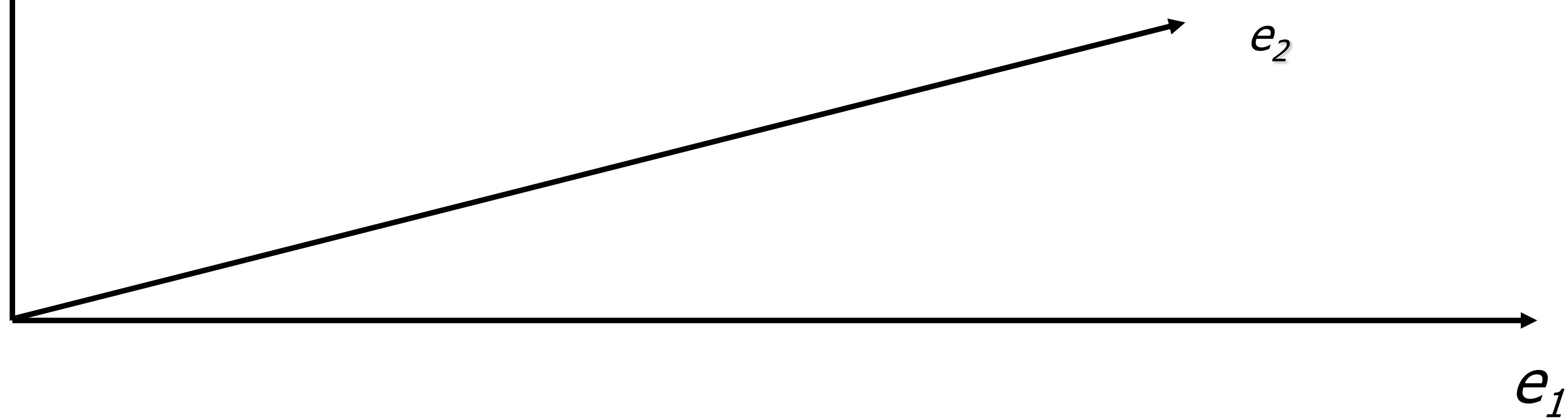


Axes non-commutative

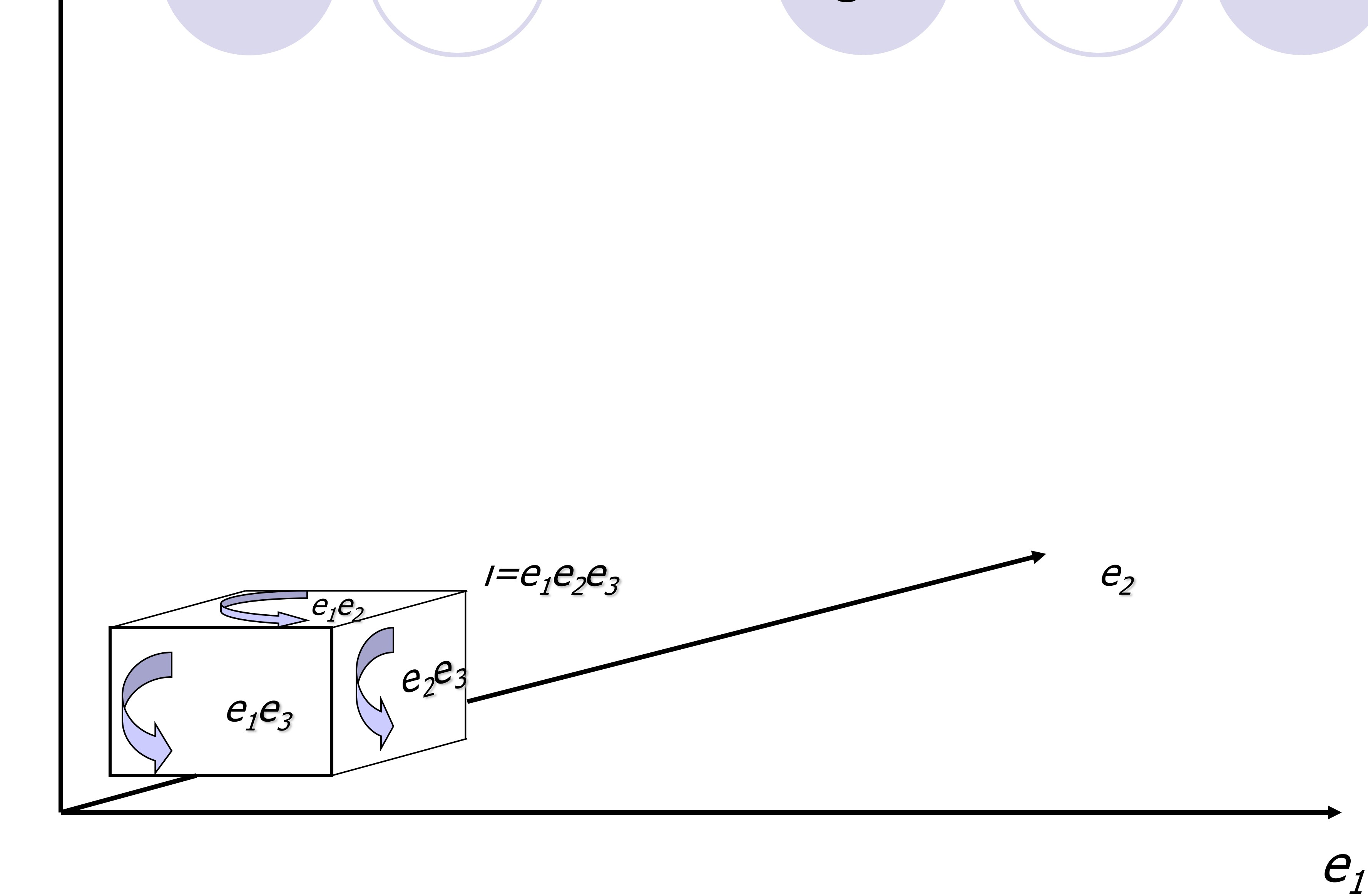
Clifford's description of space

Clifford 1879

- Adopt e_1, e_2, e_3 with $e_1^2 = e_2^2 = e_3^2 = 1$ where e_1, e_2, e_3 are viewed as simply algebraic symbols rather than unit vectors.
- Each element anti-commutes, that is $e_1 e_2 = -e_2 e_1$, $e_2 e_3 = -e_3 e_2$, *etc*
- *Define trivector* $e_1 e_2 e_3$



Clifford 3D Geometric Algebra



Timeline

2000

1878 Geometric algebra-Clifford

1843 Quaternions-Hamilton

1799 Complex numbers, Argand diagram

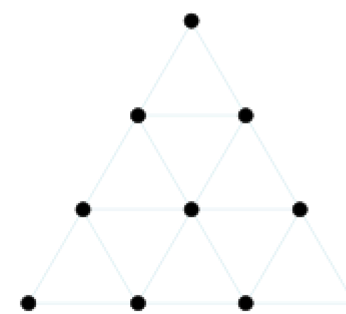
1637 Cartesian coordinates-Descarte

1545 negative numbers established, number line

1170-1250 debts seen as negative numbers-Pisa

1000

800 zero used in India



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-ve numbers used in India and China

300BC, Euclid-"Father of Geometry"

d 475BC, Pythagoras

500BC

d 547BC, Thales-"the first true mathematician"

Cliffords geometric algebra

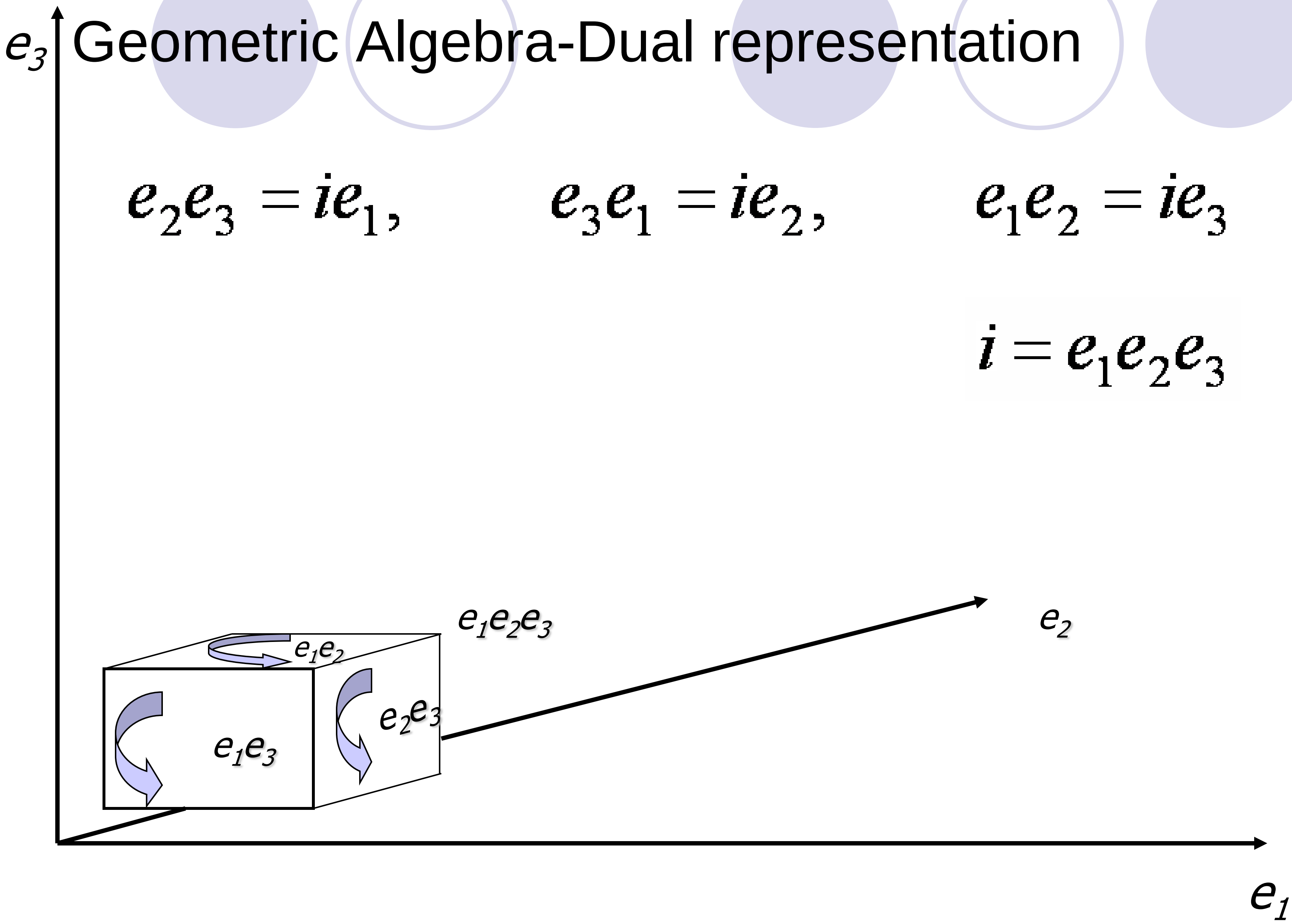
Clifford’s mathematical system incorporating 3D Cartesian coordinates, and the properties of complex numbers and quaternions into a single framework “**should have gone on to dominate mathematical physics….”,** but….

- Clifford died young, at the age of just 33
- Vector calculus was heavily promoted by Gibbs and rapidly became popular, eclipsing Clifford’s work, which in comparison appeared strange with its non-commuting variables and bilinear transformations for rotations.

Geometric Algebra-Dual representation

$$e_2 e_3 = i e_1, \qquad e_3 e_1 = i e_2, \qquad e_1 e_2 = i e_3$$

$$i = e_1 e_2 e_3$$



Multiply two vectors (expand the brackets)

Distribution of multiplication over addition)

$$
\begin{aligned}
&uv \\
&= (e_1u_1 + e_2u_2 + e_3u_3)(e_1v_1 + e_2v_2 + e_3v_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3 + (u_2v_3 - v_2u_3)e_2e_3 + (u_1v_3 - v_1u_3)e_1e_3 + (u_1v_2 - v_1u_2)e_1e_2 \\
&= u_iv_i + i[(u_2v_3 - v_2u_3)e_1 + (u_1v_3 - v_1u_3)e_2 + (u_1v_2 - v_1u_2)e_3] \\
&= u \cdot v + iu \times v
\end{aligned}
$$

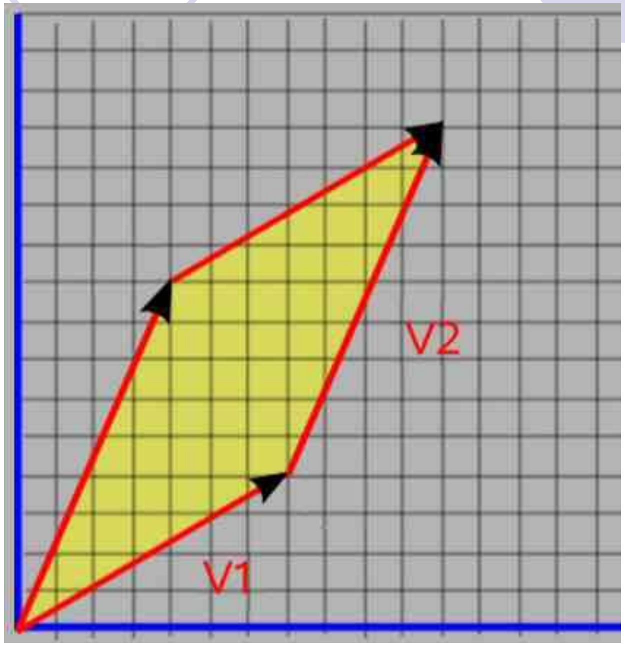
$u^2 = u \cdot u = u_1^2 + u_2^2$ a scalar.

$$i = e_1e_2e_3$$

Therefore the inverse vector is: $u^{-1} = u / u^2$ a vector with the same direction and inverse length.

The conventional dot product works for any number of dimensions and provides the metric, however the cross product is only applicable in 3D, and is an axial vector, whereas the wedge product is valid in any number of dimensions.

Areas and volumes



All dimensions in metres

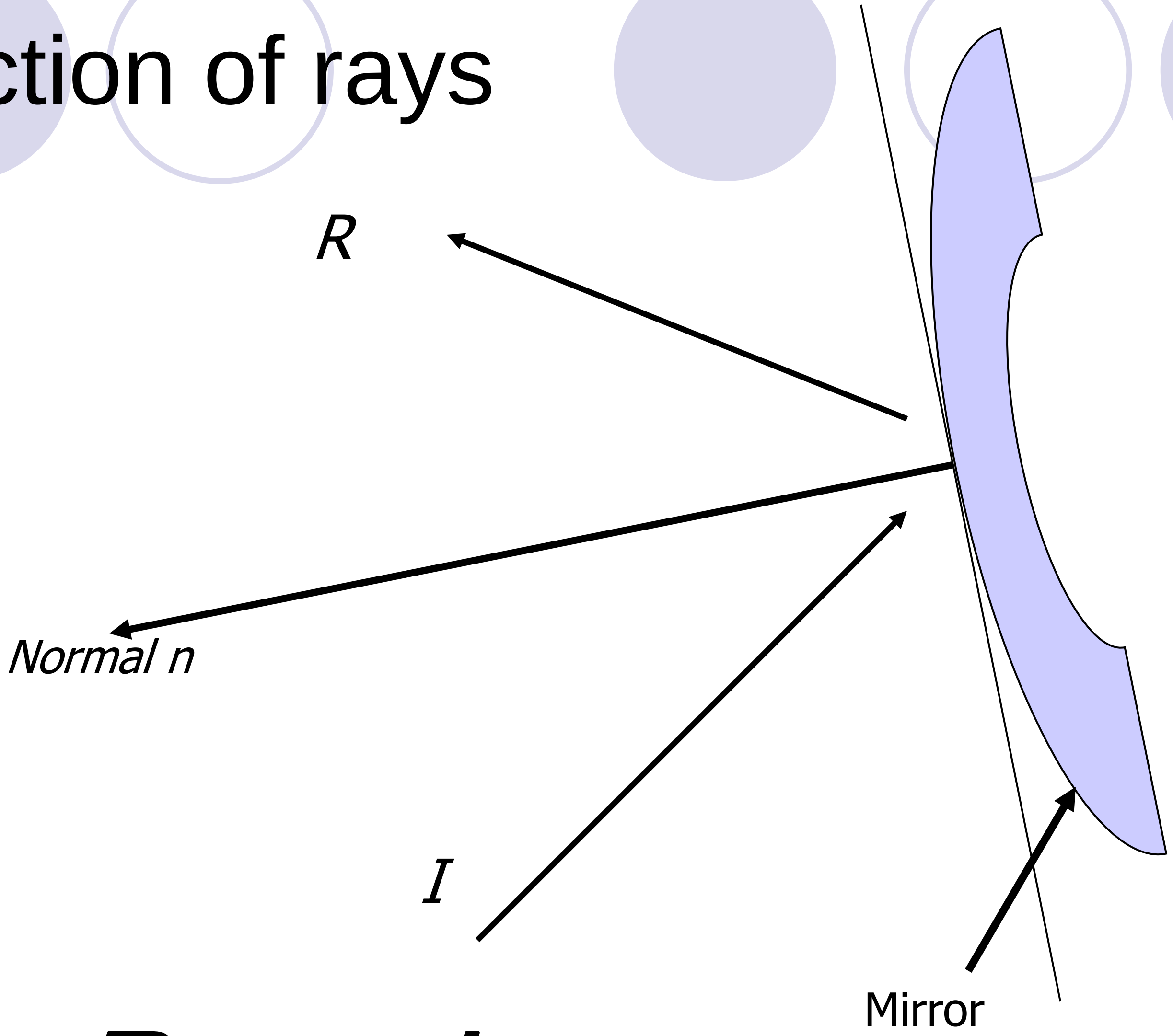
1. Calculate the enclosed area between the vectors v_1 and v_2.
$v_1=7e_1+4e_2$ and $v_2 = 4e_1+9e_2$

Ans: $A = \langle v_1\ v_2 \rangle_2\ m^2$

2. Calculate the enclosed volume between the vectors v_1 ,v_2 and v_3.
$v_1=7e_1+e_2 +4e_3$, $v_2 = 2e_1+7e_2 +5e_3$ $v_3 = e_1+3e_2 +6e_3$

Ans: $V = \langle v_1\ v_2\ v_3 \rangle_3\ m^3$

Reflection of rays



$$\boldsymbol{R} = -\boldsymbol{n}\boldsymbol{I}\boldsymbol{n}$$

So what does $j = \sqrt{-1}$ mean?

"The true metaphysics of the square root of -1 is elusive."

Gauss 1825

Imaginary numbers first appeared as the roots to quadratic equations.

For example: $x^2 + 1 = 0 \longrightarrow x = \pm\sqrt{-1}$

We find that an oriented unit area described by a bivector squares to *-1*…?

That is, in the e_1e_2 plane: $\hat{A}^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -1$

Hence the simplest geometric meaning to *i* is a **unit area**…

Eg, solve: $x^2-4x+5=0$

What is a number?

- "A number is used to represent something."

  Dr Alexander
- Real numbers(eg temperature)
- Complex numbers(waves), e.g. $5+3\,j$
- Directed number(Vectors), e.g. $2\,e_1 + 3\,e_2$
- Quaternions $3 + 2i + 3j - 4\,k$
- Is there a general type of number that can encompass all these types of numbers?

Multivector numbers($\Re \oplus \Re^3 \oplus \bigwedge^2 \Re^3 \oplus \bigwedge^3 \Re^3$)

$\bigwedge \Re^3$ Is the exterior algebra of $\Re^3$ $Cl_{3,0}(\Re)$

$M = a + v_1\, e_1 + v_2\, e_2 + v_3\, e_3 + w_1\, e_2 e_3 + w_2\, e_3 e_1 + w_3\, e_1 e_2 + b\, e_1 e_2 e_3$

$= a + \underline{v} + i\, \underline{w} + i\, b$

$z = a + i\, b$ complex numbers

$\underline{v} = v_1\, e_1 + v_2\, e_2 + v_3\, e_3$ vectors (directed numbers)

$B = i\, \underline{w} = i\, (w_1\, e_1 + w_2\, e_2 + w_3\, e_3)$ pseudovector, axial vector

$H = a + i\, \underline{w}$ quaternions, Pauli spinors

$F = \underline{E} + i\, \underline{B}$ EM field, F^2

$\psi = a + \underline{E} + i\, \underline{B} + i\, b$ *???*

Dirac spinor describing the electron

The Maths family

"The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are *nonassociative*."—John Baez

The multivector now puts the reals, complex numbers and quaternions all on an equal footing.

$M = a + \underline{v} + i\,\underline{w} + i\,b$

The leaning
tower
Of Pisa,
Italy

u.v | u x v | Matrices as basis vectors

"I skimped a bit on the foundations,
but no one is ever going to notice."

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}, \quad \text{(Gauss' law);}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E} = \mu_0 \vec{J}, \quad \text{(Ampère's law);}$$

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0, \quad \text{(Faraday's law);}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \text{(Gauss' law of magnetism)}$$

using the vector gradient:

$$\nabla = e_1 \partial_x + e_2 \partial_y + e_3 \partial_z$$

Maxwell in GA

$$uv = u \cdot v + iu \times v$$

$$\nabla \cdot E = \rho / \varepsilon$$

$$\nabla \times E + \partial_t B = 0$$

$$\nabla \times B - \partial_t E = \mu_0 J$$

$$\nabla \cdot B = 0$$

$$\xrightarrow{i}$$

$$\nabla \cdot E = \rho / \varepsilon$$

$$i\nabla \times E + \partial_t iB = 0$$

$$i\nabla \times iB + \partial_t E = -\mu_0 J$$

$$\nabla \cdot iB = 0$$

$$\swarrow$$

$$\nabla E + \partial_t iB = \frac{\rho}{\varepsilon}$$

$$\nabla iB + \partial_t E = -\mu_0 J$$

$$\longrightarrow \left(\partial_t + \nabla\right)\left(E + iB\right) = \frac{\rho}{\varepsilon} - \mu_0 J$$

Maxwell’s equation

$$\partial F = J$$

$$\partial = \partial_t + \nabla$$ Four-gradient

$$F = E + iB$$ Field variable

$$J = \mu_0 (c\rho - J)$$ Four-current

Two types of vectors….

Notationally **E** and **B** not distinguished…?

POLAR (vectors)	AXIAL (bivectors)
$\boldsymbol{E}$	$\boldsymbol{B}$
$\boldsymbol{v}$	$\boldsymbol{w} = \boldsymbol{r} \times \boldsymbol{v}$
$\boldsymbol{p} = q\, d\, \boldsymbol{r}$ (electric dipole)	$\boldsymbol{m} = I\, d\, \boldsymbol{A}$ (mag. dipole)
$\boldsymbol{f}$ (force)	$\boldsymbol{T} = \boldsymbol{r} \times \boldsymbol{F}$ (torque)

$$(\boldsymbol{p} + i\boldsymbol{m})\, F = (\boldsymbol{p} + i\boldsymbol{m})(\boldsymbol{E} + i\, \boldsymbol{B}) = -U + i\boldsymbol{T}$$

Electromagnetic waves

We can define a plane wave

$$F = F_0 e^{\pm i(\mathbf{k}\cdot\mathbf{r} - wt)} \qquad F_0 = E_0 + iB_0$$

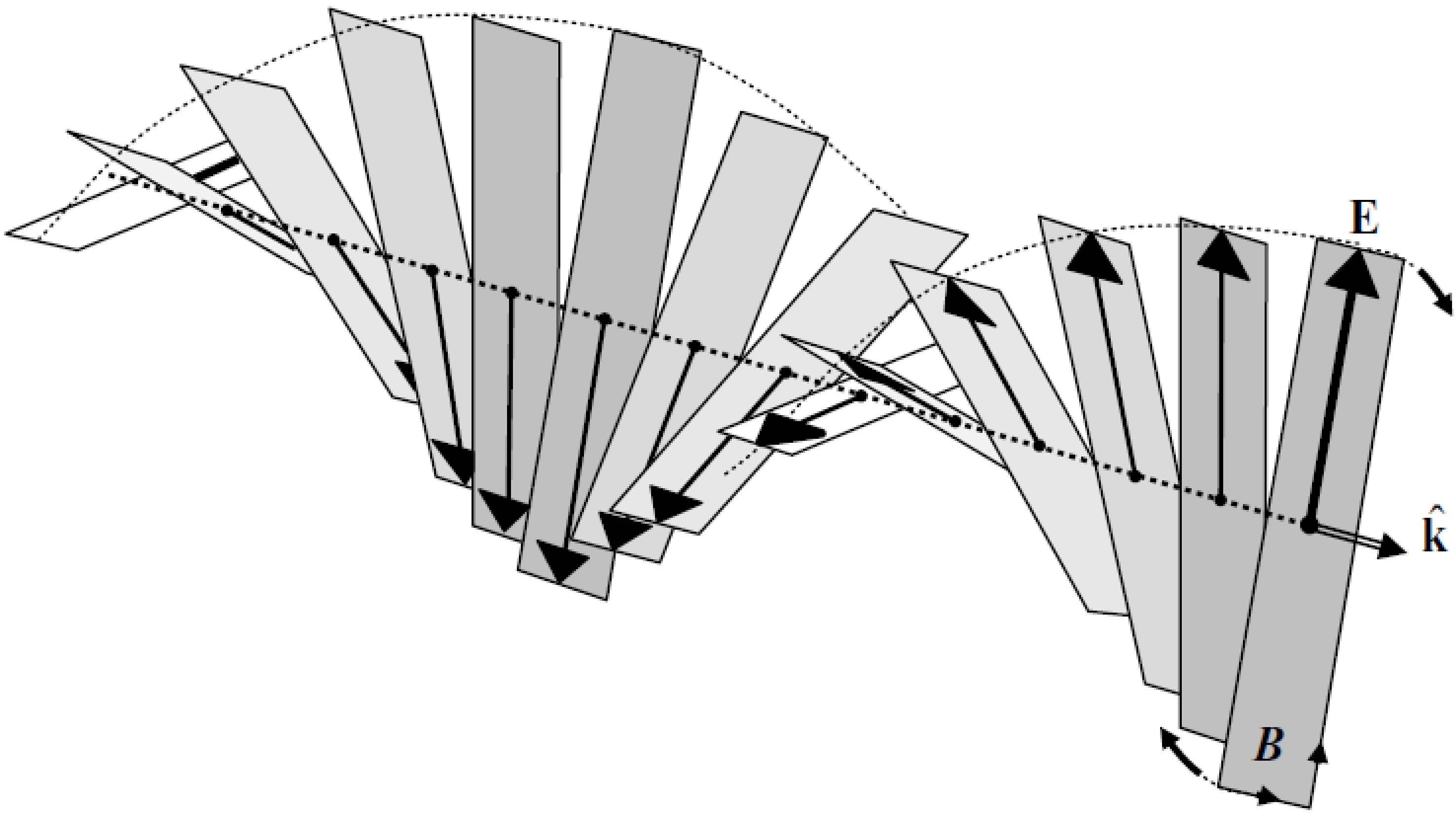
$$(\partial_t + \nabla) F = \pm i (w - \mathbf{k}) F = 0 \qquad \nabla e^{-i\mathbf{k}\cdot\mathbf{r}} = -i\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}}$$

$$\Rightarrow \ (w - \mathbf{k})(\mathbf{E}_0 + i\mathbf{B}_0) = 0$$

$$\Rightarrow \ \underbrace{-\mathbf{k}\cdot\mathbf{E}_0}_{\text{scalar}} + \underbrace{w\mathbf{E}_0 - i\mathbf{k}\wedge\mathbf{B}_0}_{\text{vector}} - \underbrace{\mathbf{k}\wedge\mathbf{E}_0 - wi\mathbf{B}_0}_{\text{bivector}} - \underbrace{i\mathbf{k}\cdot\mathbf{B}_0}_{\text{trivector}} = 0$$

Circular polarization by default

$$F = F_0 e^{\pm i(\mathbf{k}\cdot\mathbf{r} - wt)}$$



Potential formulation

Assuming: $F = (\partial_t - \nabla) A$

$$(\partial_t + \nabla) F = (\partial_t + \nabla)(\partial_t - \nabla) A$$

$$\Rightarrow \left(\partial_t^2 - \nabla^2\right) A = J$$

So we require a multivector potential

$A = V - \mathbf{A}$ to correspond to a source

$J = \rho - \mathbf{J}$

Special relativity

Its easiest to begin in 2D, which is sufficient to describe most phenomena.
We define a 2D spacetime event as $\boldsymbol{x} = x_1 e_1 + x_2 e_2$ $\quad i=e_1e_2$

$$X = x + it$$

So that time is represented as the bivector of the plane and so an extra Euclidean-type dimension is not required.

We find: $X^2 = x^2 - t^2$ the correct spacetime distance.

We have the general Lorentz transformation given by:

$$X' = e^{-\phi\hat{v}/2} e^{-i\theta/2} X e^{i\theta/2} e^{\phi\hat{v}/2}$$

Consisting of a rotation and a boost, which applies uniformly to both coordinate and field multivectors.

$$P' = -\hat{\mathbf{v}} \mathrm{e}^{-\phi\hat{\mathbf{v}}/2} P \mathrm{e}^{\phi\hat{\mathbf{v}}/2} \hat{\mathbf{v}}$$

Compton scattering formula

C

A few negatives

$$e^{i\pi} = -1$$

"…it is absolutely paradoxical; we cannot understand it, and we don't know what it means."
Benjamin Peirce

Setting $i = e_1 e_2$ we have in 2D $e^{i\pi/2}\vec{v}e^{-i\pi/2} = -\vec{v}$

We can view the formula above as describing a rotation that flips a vector in sign, but

$$e^{i\pi/2}\vec{v}e^{-i\pi/2} = -\vec{v}$$

$$\Rightarrow e^{i\pi}\vec{v} = -\vec{v}$$

i anti-commutes in 2D

$$\Rightarrow e^{i\pi} = -1$$

i.e. **rotating a vector by π flips the sign**

Foundational errors in mathematical physics

1. By not recognizing that the vector dot and cross products are two halves of a single combined geometric product.
   Circa 1910.
2. That the non-commuting properties of matrices are a clumsy substitute for Clifford’s non-commuting orthonormal axes of three-space.
   Circa 1930.

Research areas in GA

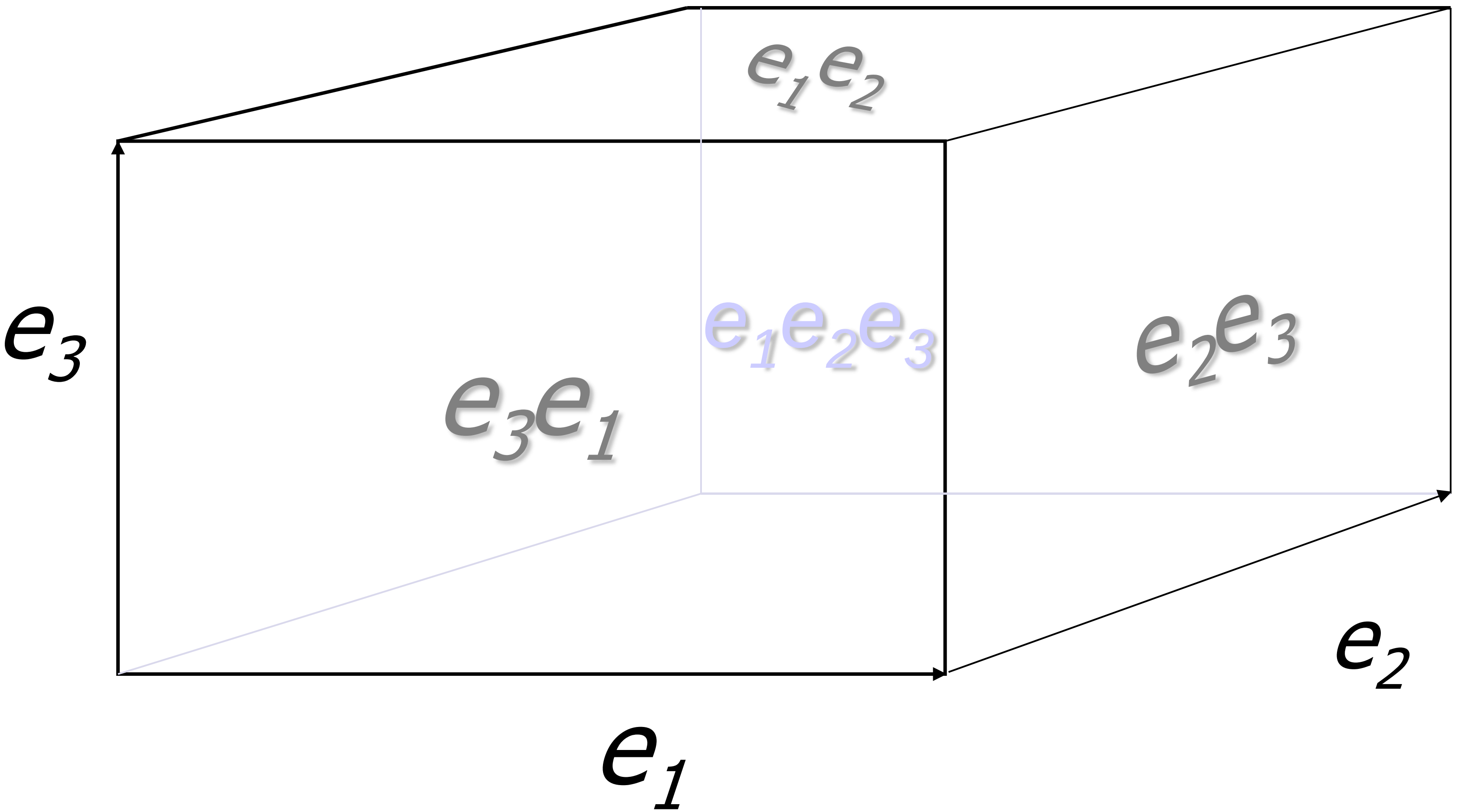
- black holes and cosmology
- quantum tunneling and quantum field theory
- beam dynamics and buckling
- computer vision, computer games
- quantum mechanics-EPR
- quantum game theory
- signal processing-rotations in N dimensions, wedge product also generalizes to N dimensions

$$uv = u \cdot v + iu \times v$$

Conclusion

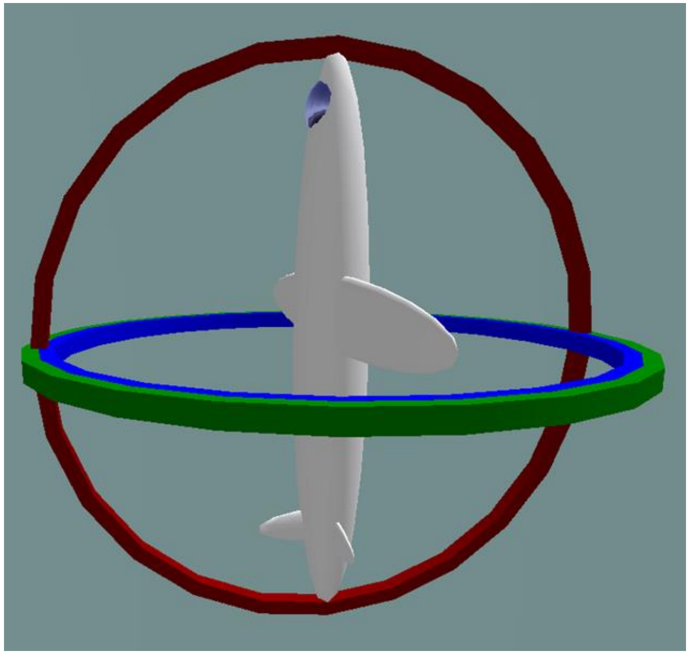
- Geometric algebra unifies complex numbers, quaternions, vectors, axial vectors into a single multivector
- Useful in describing Maxwells equations, RLC circuits, EM waves, anisotropic media including metamaterials and general relativity.

Algebraic description of space



Use of quaternions

Used in airplane guidance systems to avoid Gimbal lock



Quotes

- “The reasonable man adapts himself to the world around him. The unreasonable man persists in his attempts to adapt the world to himself. Therefore, all progress depends on the unreasonable man.” George Bernard Shaw,
- Murphy’s two laws of discovery:

  “All great discoveries are made by mistake.”

  “If you don't understand it, it's intuitively obvious.”
- “It's easy to have a complicated idea. It's very hard to have a simple idea.” Carver Mead.

Greek concept of the product
Euclid Book VII(B.C. 325-265)

“1. A **unit** is that by virtue of which each of the things that exist is called one.”

“2. A **number** is a multitude composed of units.”

….

“16. When two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another.”

How many space dimensions do we have?

- The existence of five regular solids implies three dimensional space(6 in 4D, 3 > 4D)
- Gravity and EM follow inverse square laws to very high precision. Orbits(Gravity and Atomic) not stable with more than 3 D.
- Tests for extra dimensions failed, must be sub-millimetre

What is time?

- “Of all obstacles to a thoroughly penetrating account of existence, none looms up more dismayingly than time.” Wheeler 1986
- In GA time is a bivector, ie rotation.
- Clock time(EM), Dynamical time(Gravity) and Entropy arrow of time
- Space=3 translational freedoms,
  Time = 3 rotational freedoms of physical space.

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References

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- **A New Framework Based on Geometric Algebra for the Analysis of Materials and Metamaterials with Electric and Magnetic Anisotropy** , S.A. Matos, C.R. Paiva, and A.M. Barbosa, Antennas and Propagation society international Symposium 2008. AP-S 2008 IEEE.

Reminders on the Board

$$j = \sqrt{-1}$$

$$\vec{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$$

$$i = e_1 e_2 e_3$$

$$M = a + \vec{v} + i\vec{w} + ib$$

$$ie_1 = e_2 e_3, ie_2 = e_3 e_1, ie_3 = e_1 e_2$$

$$i\vec{w} = i\left(w_1 e_1 + w_2 e_2 + w_3 e_3\right) = w_1 e_2 e_3 + w_2 e_3 e_1 + w_3 e_1 e_2$$

Substitutions:

$$B \to cB, A \to cA, J \to c\mu J, \rho \to \rho / \varepsilon_0, \partial_t \to \partial_{ct}, k \to ck$$

General solution to Maxwell in potential form

$$V(\mathbf{r},t) = \frac{1}{4\pi}\int \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}d\tau'\ ,\ A(\mathbf{r},t) = \frac{1}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}d\tau'$$

using retarded times.

These can be combined in geometric algebra:

$$A = V - \mathbf{A} = \frac{1}{4\pi}\int \frac{J(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \qquad J = \rho - \mathbf{J}$$

The field is then calculated from

$$F = (\partial_t - \nabla)\, A \qquad F = E + \imath B$$

Gauge freedom(optional)

Enforcing the Lorenz gauge

$$\partial_t V + \nabla \cdot \mathbf{A} = 0$$

Without affecting **E** and **B**, or the Lorenz gauge we can make the substitution:

$$V' = V - \partial_t \lambda\,,\; A' = A + \nabla \lambda$$

provided

$$\nabla^2 \lambda - \partial_t^2 \lambda = 0$$

Field properties

Multiplying from the right with the four gradient

$$(\partial_t - \nabla)(\partial_t + \nabla) F = 0$$

$$\Rightarrow \left(\partial_t^2 - \nabla^2\right) F = 0$$

$$\Rightarrow \left(\partial_t^2 - \nabla^2\right)(\mathbf{E}_0 + i\mathbf{B}_0) = 0$$

Therefore $\mathbf{E}_0$ and $\mathbf{B}_0$ separately satisfy the 3D wave equation.

Other references

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A simply Ball model for kids

Simplify

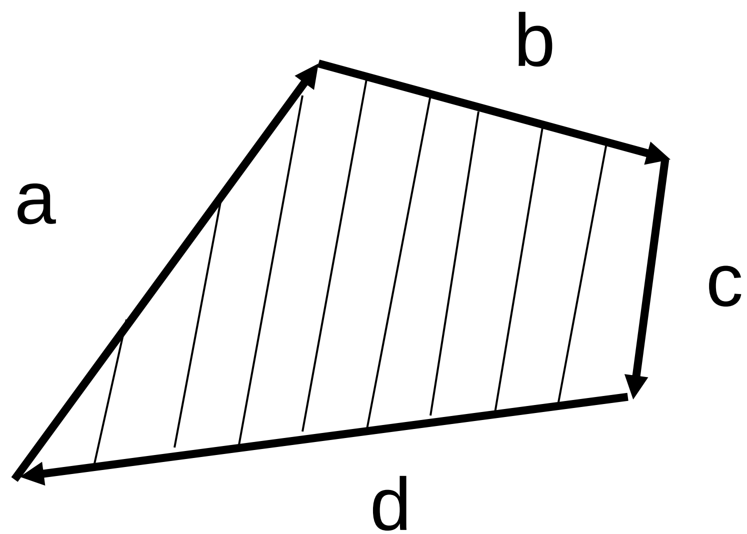
$$(2e_1)(3e_3)2e_1 = 12e_1e_3e_1 = -12e_3$$

Simplify

$$(3e_3e_1 + 2e_1)2e_2 = 6e_1e_2e_3 + 4e_1e_2$$

Enclosed area of polygons

All dimensions in metres



$$A=\tfrac{1}{2}\langle ab+bc+cd+da\rangle_2 \; m^2$$

Talks

- Why study it? What problems like to solve?
- Each slide one idea
- Take home msg

RLC circuit analysis

An applied voltage: $V = V_m e_1 \mathrm{e}^{\iota(wt+\theta)}$ $\iota = e_1 e_2$

$$I = I_m e_1 \mathrm{e}^{\iota(wt+\theta)} = I_m e_1 \mathrm{e}^{\iota\theta} \mathrm{e}^{\iota wt} = I_m(\cos\theta e_1 + \sin\theta e_2)\mathrm{e}^{\iota wt}$$

The potential across an inductor: $V_L = L\frac{dI}{dt} = LI\iota w$

For a RL series circuit we have

$$\begin{aligned} V &= V_R + V_L \\ &= (R - \iota wL)I \end{aligned}$$

Lorenz gauge

Using $F = (\partial_t - \nabla) A$

we find:

$$F = (\partial_t - \nabla)(V - \mathbf{A})$$

$$\mathbf{E} + i\mathbf{B} = \underbrace{\partial_t V + \nabla \cdot \mathbf{A}}_{\text{scalar}} + \underbrace{-\nabla V - \partial_t \mathbf{A}}_{\text{vector}} + \underbrace{\nabla \wedge \mathbf{A}}_{\text{bivector}}$$

We require $\partial_t V + \nabla \cdot \mathbf{A} = 0$

which is the Lorenz gauge.

This gauge uniquely places V and A on equal footing.